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Stability of Orthogonally Quintic Functional Equation in Multi-Banach Spaces

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Abstract. In this paper, we establish the Hyers-Ulam stability of the orthogonally quintic functional equation in Multi-Banach Spaces.

Key words: Hyers-Ulam stability, Multi-Banach Spaces, orthogonally quintic functional equation, Fixed Point Method.

AMS Subject classification: Primary 39B82, 39B52, 47H10, 46H25

1. Introduction

The stability problem for functional equations starts from the famous talk of Ulam and the partial solution of Hyers to the Ulam's problem see ([17] and [6]). Thereafter, Rassias [14] attempted to solve the stability problem of the cauchy additive functional equation in a more general setting.

The concept introduced by Rassias's theorem significantly influenced a number of mathematicians to investigate the stability problems for various functional equations see ([1], [3], [6], [7], [8], [9], [10], [16], [20]).

In 2013, Fridoun Moradlou [5] proved the generalized Hyers-Ulam-Rassias stability of the Euler-Lagrange-Jensen Type Additive mapping in Multi-Banach Spaces. In 2015, Xiuzhong Yang, Lidan Chang, Guofen Liu [19] estabilished the orthogonal stability of mixed additive-quadratic jensen type functional equation in Multi-Banach Spaces. In 2016, Sattar Alizadeh, Fridoun Moradlou [15] proved the generalized Hyers-Ulam-Rassias stability of the quadratic mapping in multi-Banach spaces.

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Choonkil Park, Jian Lian CUI, Madjid Eshaghi GORDJI [2], proved the Hyers-Ulam stability of an orthogonally quintic functional equation in Banach Spaces.

Theorem 1.1. *[13] Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(T^n, T^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer m such that

(1) $d(T^n, T^{n+1}x) < \infty, \forall n \geq m$;

(2) *the sequence $\{T^n x\}$ converges to a fixed point u^* of T;*

(3) *u^* is the unique fixed point of T in the set $Y = \{u \in X : d(T^m x, u) < \infty\}$;*

(4) $d(u, u^*) \leq \dfrac{1}{1-\alpha} d(u, Tu)$ *for all $u \in Y$.*

Definition 1.2. [4] A Multi- norm on $\{\wp^k : k \in \mathbb{N}\}$ is a sequence $(\|.\|) = (\|.\|_k : k \in \mathbb{N})$ such that $\|.\|_k$ is a norm on $\wp^k$ for each $k \in \mathbb{N}$, $\|x\|_1 = \|x\|$ for each $x \in \wp$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

(1) $\left\|\left(x_{\sigma(1)} ... x_{\sigma(k)}\right)\right\|_k = \|(x_1 ... x_k)\|_k$, for $\sigma \in \Psi_k, x_1 ... x_k \in \wp$;

(2) $\|\alpha_1 x_1 ... \alpha_k x_k\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|) \|(x_1 ... x_k)\|_k$ for $\alpha_1 ... \alpha_k \in \mathbb{C}, x_1, ..., x_k \in \wp$;

(3) $\|(x_1, ..., x_{k-1}, 0)\|_k = \|(x_1, ... x_{k-1})\|_{k-1}$, for $x_1, ... x_{k-1} \in \wp$;

(4) $\|(x_1 ... x_{k-1}, x_{k-1})\|_k = \|(x_1 ... x_{k-1})\|_{k-1}$ for $x_1 ... x_{k-1} \in \wp$.

In this case, we say that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - normed space.

Suppose that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - normed spaces, and take $k \in \mathbb{N}$. We need the following two properties of multi - norms. They can be found in [4].

$$(a)\, \|(x, ... x)\|_k = \|x\|, \text{for} x \in \wp,$$

$$(b) \max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, ..., x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\|, \text{for} x_1, ..., x_k \in \wp.$$

It follows from (b) that if $(\wp, \|.\|)$ is a Banach space, then $(\wp^k, \|.\|_k)$ is a Banach space for each $k \in \mathbb{N}$; In this case, $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - Banach space.

Definition 1.3. [12] Suppose that X is a vector space (algebraic module) with $\dim X \geq 2$, and $\perp$ is a binary relation on X with the following properties:

(1) Totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;

(2) Independence: If $x, y \in X - 0$ and $x \perp y$, then x and y are linearly independent;

(3) Homogeneity: If $x, y \in X$ and $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(4) Thalesian properity: If P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$ which is the set of non-negative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space (resp., module). By an orthogonality normed space (normed module) we mean an orthogonality space (resp., module) having a normed (resp., normed module) structure.

In this paper, we achieve the Hyers - Ulam stability in orthogonally quintic functional equation in Multi-Banach Spaces

$$Df(x,y) = f(3x+y) - 5f(2x+y) + f(2x-y) + 10f(x+y) - 5f(x-y)$$
$$-10f(y) - f(3x) + 3f(2x) + 27f(x). \quad (1.1)$$

Theorem 1.4. *Let X be an orthogonality space and let $\big((Y^k, \|.\|) : K \in \mathbb{N}\big)$ be a multi-Banach Suppose that β is a nonnegative real number and $f : X \to Y$ is a mapping satisfying*

$$\sup_{k \in \mathbb{N}} \|(Df(x_1, y_1), ..., Df(x_k, y_k))\|_k \leq \beta \quad (1.2)$$

$x_1, ..., x_k, y_1, ..., y_k \in P$ and $x_i \perp y_i$ $(i = 1, 2...k)$ and $f(0) = 0$. Then there exists a unique orthogonally quintic mapping $Q_5 : X \to Y$ such that

$$\sup_{k \in \mathbb{N}} \|(f(x_1) - Q_5(x_1), ..., f(x_k) - Q_5(x_k))\|_k \leq \frac{1}{31}\beta \quad (1.3)$$

$x_1, x_2, ..., x_k \in X$.

Proof. Letting $y_1 = y_2 =, ..., = y_k = 0$ in (1.2), we obtain that

$$\sup_{k \in \mathbb{N}} \|(32f(x_1) - f(2x_1), ..., 32f(x_k) - f(2x_k))\| \leq \beta \quad (1.4)$$

for all $x_1, ..., x_k \in X, x_i \perp 0 \quad \text{where}(i = 1, 2, ..., k)$.

Dividing on both side by 32 in (1.4), we get

$$\sup_{k\in\mathbb{N}} \left\| \left(f(x_1) - \frac{1}{32}f(2x_1), ..., f(x_k) - \frac{1}{32}f(2x_k) \right) \right\| \leq \frac{1}{32}\beta \tag{1.5}$$

Let $\Lambda = \{g : P \to Q | g(0) = 0\}$ and introduce the generalized metric d defined on λ by

$$d(l,m) = \inf \left\{ \lambda \in [0,\infty] | \sup_{k\in\mathbb{N}} \|l(x_1) - m(x_1), ..., l(x_k) - m(x_k)\|_k \leq \lambda \quad \forall x_1, ..., x_k \in X \right\}$$

Then it is easy to show that Λ, d is a generalized complete metric space. See [11]. We define an operator $\mathcal{J} : P \to P$ by

$$\mathcal{J}l(x) = \frac{1}{32}l(2x) \qquad x \in X.$$

We assert that $\mathcal{J}$ is a strictly contractive operator. Given $l, m \in \Lambda$, let $\lambda \in [0,\infty]$ be an arbitary constant with $d(l,m) \leq \lambda$. From the definition d, if follows that

$$\sup_{k\in\mathbb{N}} \|l(x_1) - m(x_1), ..., l(x_k) - m(x_k)\|_k \leq \lambda \quad x_1, ..., x_k \in X.$$

Therefore,

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(\mathcal{J}l(x_1) - \mathcal{J}m(x_1), ..., \mathcal{J}l(x_k) - \mathcal{J}m(x_k))\|_k \\
&\qquad \leq \sup_{k\in\mathbb{N}} \left\| \left(\frac{1}{32}l(2x_1) - \frac{1}{32}m(2x_1), ..., \frac{1}{32}l(2x_k) - \frac{1}{32}m(2x_k) \right) \right\|_k \\
&\qquad \leq \frac{1}{32}\lambda
\end{aligned}$$

$x_1, ..., x_k \in X$.

Hence,it holds that

$$d(\mathcal{J}l, \mathcal{J}m) \leq \frac{1}{32}\lambda d(\mathcal{J}l, \mathcal{J}m) \leq \frac{1}{32}d(l,m) \qquad \forall l, m \in \Lambda.$$

This Means that $\mathcal{J}$ is strictly contractive operator on Λ with the Lipschitz constant $L = \dfrac{1}{32}$.

By (1.5), we have $d(\mathcal{J}f, f) \leq \dfrac{1}{32}\beta < \infty$. According to Theorem 1.1, we deduce the existence of a fixed point of $\mathcal{J}$ that is the existence of mapping $Q_5 : P \to Q$ such that

$$Q_5(2x) = 32Q_5(x) \qquad \forall x \in X.$$

Moreover, we have $d\left(\mathcal{J}^n f, Q_5\right) \to 0$, which implies

$$Q_5(x) = \lim_{n\to\infty} \mathcal{J}^n f(x) = \lim_{n\to\infty} \frac{f(2^n x)}{32^n}$$

for all $x \in X$.
Also, $d(f, Q_5) \leq \dfrac{1}{1-L} d(\mathcal{J}f, f)$ implies the inequality

$$\begin{aligned} d(f, Q_5) &\leq \frac{1}{1-\dfrac{1}{32}} d(\mathcal{J}f, f) \\ &\leq \frac{1}{31}\beta. \end{aligned}$$

Considering Definition, we have $2^n x \perp 2^n y$. Set $x_1 =, \cdots, = x_k = 2^n x$, $y_1 =, \cdots, = y_k = 2^n y$ in (1.2) and divide both sides by 32^n. Then, using property (a) of multi-norms, we obtain

$$\begin{aligned} \|DQ_5(x,y)\| &= \lim_{n\to\infty} \frac{1}{32^n} \|Df\left(2^n x, 2^n y\right)\| \\ &\leq \lim_{n\to\infty} \frac{\beta}{32^n} = 0 \end{aligned}$$

for all $x, y \in X$. Hence Q_5 is Quintic.
The uniqueness of Q_5 follows from the fact that Q_5 is the unique fixed point of $\mathcal{J}$ with the property that there exists $\ell \in (0, \infty)$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - Q_5(x_1), ..., f(x_k) - Q_5(x_k))\|_k \leq \ell$$

for all $x_1, ..., x_k \in X$.
This completes the proof of the Theorem. $\square$

Theorem 1.5. *Let $\phi : X^{2k} \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\phi(x_1, y_1, .., x_k, y_k) \leq 32\alpha\phi\left(\frac{x_1}{2}, \frac{y_1}{2}, .., \frac{x_k}{2}, \frac{y_k}{2}\right) \tag{1.6}$$

for all $x_i, y_i \in X$ with $x_i \perp y_i$, where $i = 1, .., k$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|Df(x_1, y_1, .., x_k, y_k)\| \leq \phi(x_1, y_1, .., x_k, y_k) \tag{1.7}$$

for all $x_i, y_i \in X$ *with* $x_i \perp y_i$, *where* $i = 1, .., k$. *Then there exists a unique orthogonally quintic mapping* $Q_5 : X \to Y$ *such that*

$$\|(f(x_1) - Q_5(x_1), ..., f(x_k) - Q_5(x_k))\| \leq \frac{\alpha}{1-\alpha}\phi(x_1, 0, ..., x_k, 0) \tag{1.8}$$

for all $x_i \in X$, *where* $i = 1, .., k$.

Proof. Taking $y_i = 0$ in (1.7), we get

$$\|(32f(x_1) - f(2x_1), .., 32f(x_k) - f(2x_k))\| \leq \phi\left(x_1, 0, .., x_k, 0\right) \tag{1.9}$$

for all $x_i \in X$, since $x_i \perp 0$, where $i = 1, .., k$. So

$$\left\|\left(f(x_1) - \frac{1}{32}f(2x_1), ..., f(x_k) - \frac{1}{32}f(2x_k)\right)\right\| \leq \alpha\phi\left(x_1, 0, .., x_k, 0\right) \tag{1.10}$$

for all $x_i \in X$, where $i = 1, .., k$. Consider the set $G : h : X \to Y$ and introduce the generalized metric on G.

$$d(g, h) = inf\left\{\mu \in \mathbb{R}_+ : \|(g(x_1) - h(x_1), .., g(x_k) - h(x_k))\| \leq \mu\phi\left(x_1, 0, .., x_k, 0\right) \quad \forall x_i \in X\right\}$$

where $i = 1, .., k$. It is easy to prove that (G, d) is complete.See [11]. It follows from (1.10) that $d(f, Jf) \leq \alpha$. The rest of the proof is similar to the proof of Theorem 1.1. □

Corollary 1.6. *Let* θ *be a positive real number and* p *a real number with* $p > 5$. *Let* $f : X \to Y$ *be a mapping satisfying*

$$\|(Df(x_1, y_1, .., x_k, y_k))\| \leq \theta\left(\|x_1\|^p + \|y_1\|^p, ..., \|x_k\|^p + \|y_k\|^p\right) \tag{1.11}$$

for all $x_i, y_i \in X$, *since* $x_i \perp y_i$, *where* $i = 1, .., k$. *Then there exists a unique orthogonally quintic mapping* $Q_5 : X \to Y$ *such that*

$$\|(f(x_1) - Q_5(x_1), ..., f(x_k) - Q_5(x_k))\| \leq \frac{2^p\theta}{32 - 2^p}\left(\|x_1\|^p, ..., \|x_k\|^p\right)$$

for all $x_i \in X$, *where* $i = 1, .., k$.

Proof. The proof follows from Theorem1.5 by taking $\phi(x_1, y_1, .., x_k, y_k) = \theta\left(\|x_1\|^p + \|y_1\|^p, .., \|x_k\|^p + \|y_k\|^p\right)$ for all $x_i, y_i \in X$, since $x_i \perp y_i$, where $i = 1, .., k$. Then we can choose $\alpha = 2^{p-5}$ and we get the desired result. □

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