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Controllability of a Model On Deforestation Due to Human Population and its Effect on Farm Fields

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Abstract. In this paper, we propose a mathematical model on deforestation due to human population using a system of stochastic difference equations. Then we include the control variable to the system of difference equations. First, we analyze the controllability of the proposed model. We can see that the desired objective can be achieved by manipulating the chosen control variables. And then we analyze that the model is controllable at the origin.

Key words: Stochastic difference Equations, Controllability, Control variables, Deforestation.

1. Introduction

Human ecology has been defined as a type of analysis applied to the relations in human beings that was traditionally applied to plants and animals in ecology. Deforestation, is the removal of a forest or stand of trees where the land is thereafter converted to a non-forest use. Forests cover almost a third of the earth's land surface. Forests provides many environmental benefits including its major role in the hydrologic cycle, soil conservation, prevention of climate change and preservation of biodiversity.

Trees are highly effective in absorbing water quantities, keeping the amount of water in watersheds to a manageable level. The forest also serves as cover against erosion. Forests perform a valuable function by capturing rainwater and releasing

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it to streams and rivers that provide water for cities and agriculture. Forest soils with a carpet of decomposing leaves absorb rainwater like a sponge, holding the water for gradual release to streams throughout the year. When watersheds lose their forest, the soil can lose its capacity to absorb rainwater as it did before. Rainwater flows quickly off the watershed, causing floods during the rainy season and a diminished supply of water during the dry season.

In addition, the flow of water from the hills irrigate the farm fields. And the quality of the water is worse because deforested hills no longer have trees to protect the ground from heavy rain, so soil erosion is greater, and the irrigation water contains large quantities of mud that settles in irrigation canals and clogs the canals. This decline in the quantity and quality of irrigation water reduces food production even further. The result is poor nutrition and health for people. This chain of effects involving human population growth, deforestation and lower food production is a vicious cycle that is difficult to escape. Forests benefits all who live downstream by reducing erosion and flooding. Floods and erosion will cause severe constraints on food production throughout the world. Farmers in villages depend on the sponge-effect of forests to absorb and slowly release water. Many agricultural exports, are dependent upon forest-generated soils and water[6].

In the last three decades, control theory has gained importance as a discipline for engineers, mathematicians, scientists, and other researchers. The theory of controllability and observability has been developed, in response to problems generated by technological science, especially in areas related to control, communication, and computers. Controllability is an important property of a control system, and the controllability property plays a crucial role in many control problems, such as stabilization of unstable systemsby feedback, or optimal control. Controllability and observability represent two major concepts of modern control system theory. These concepts were introduced by R. Kalman in 1960. We are mainly interested in the problem of whether it is possible to steer a system from a given initial state to any arbitrary state in a finite time period[5]. In order to be able to do whatever we want with the given dynamic system under control input, the system must be controllable. Similarly, in order to see what is going on inside the system under observation, the system must be observable.

In Section 2, we provide the preliminaries on controllability of a discrete system. In Section 3, we formulate a model on deforestation by human population and its effect on farm fields using stochastic difference equations and introduce the control variable to the system. In Section 4, we have analyzed the controllability of the model and then study the controllability at the origin.

2. Preliminaries and Theorems

Definition 2.1. Controlled system

Consider a controlled system given by

$$x(n+1) = Ax(n) + Bu(n) \tag{1}$$

Where A is a $k \times k$ matrix, B is a $k \times m$ matrix called input matrix and $u(n)$is a $m \times 1$vector also known as control variable.

Definition 2.2. Completely Controllable

The system (1) is said to be completely controllable or simply controllable if for any $n_0 \in Z^+$, any intial state $x(n_0) = x_0$, and any given final state x_f, there exists a finite $N > n_0$ and a control $u(n)$, $n_0 < n \leq N$, such that $x(N) = x_f$.

Definition 2.3. Controllability Matrix

The controllability matrix of system (1) is defined by $k \times km$matrix given by

$$W = [B, AB, A^2B, ...A^{k-1}B]$$

The controllability matrix plays a major role in control theory, as may be seen in the following important basic result.

Definition 2.4. Controllability to the Origin

A system is controllable to the origin if, for any $n_0 \in Z^+$ and $x_0 \in \Re^k$, there exists a finite $N > n_0$ and a control $u(n)$, $n_0 < n \leq N$, such that $x(N) = 0$.

Theorem 2.5. ***Theorem 1.*** *System* (1) *is completely controllable if and only if rank* $W = k$ *[5].*

3. Controlled System for Deforestation

Let $H(t), B(t)$and $F(t)$denote the human population, density of trees in the forests, density of crops in the farm fields at time t respectively. We are considering a three dimensional discrete stochastic process[7].

$$X(t) = [H(t), B(t), F(t), T(t)]^T \tag{2}$$

Table 1. Notation for transitions of the Deforestation due to human population and its effect on farm fields

S.No	Transition	Transition Probability	Transition Number
1	$H \to B$	$\beta_1(t) = 0$	$B_1(t) = 0$
2	$B \to H$	$\alpha_1(t)$	$F_1(t)$
3	$B \to F$	$\beta_2(t)$	$B_2(t)$
4	$F \to B$	$\alpha_2(t) = 0$	$F_2(t) = 0$
5	$F \to H$	$\gamma_1(t)$	$G_1(t)$
6	$H \to F$	$\gamma_2(t)$	$G_2(t)$
7	$H \to T$	$\mu_1(t)$	$M_1(t)$
8	$T \to B$	$\mu_2(t)$	$M_2(t)$
9	$H \to$ Death	$d_H(t)$	$D_H(t)$
10	$B \to$ Death	$d_B(t)$	$D_B(t)$
11	$F \to$ Death	$d_F(t)$	$D_F(t)$
12	$T \to$ Death	$d_T(t)$	$D_T(t)$
13	immigration $\to H$	$\mu_H(t)$	$R_H(t)$

We obtain the following Stochastic Difference Equations:

$$\begin{aligned}
H(t+1) &= H(t) + R_H(t) - D_H(t) + F_1(t) + G_1(t) - G_2(t) \\
B(t+1) &= B(t) - F_1(t) - D_B(t) - B_2(t) + M_2(t) \\
F(t+1) &= F(t) + B_2(t) + G_2(t) - G_1(t) \\
T(t+1) &= T(t) + M_1(t) - M_2(t)
\end{aligned} \tag{3}$$

We rewrite the above system (3) as follows

$$
\begin{aligned}
H(t+1) &= [1+\mu_H(t)-\gamma_2(t)-d_H(t)]\, H(t)+\alpha_1(t)B(t)+\gamma_1(t)F(t)\\
B(t+1) &= [1-\alpha_1(t)-\beta_2(t)-d_B(t)]\, B(t)+\mu_2(t)T(t)\\
F(t+1) &= [1-\gamma_1(t)-d_F(t)]F(t)+\gamma_2(t)H(t)+\beta_2(t)B(t)\\
T(t+1) &= [1-\mu_2(t)-d_T(t)]T(t)+\mu_1(t)H(t)
\end{aligned}
\tag{4}
$$

Now we formulate a controlled system for the above system (4)

$$
\begin{aligned}
H(t+1) &= [1+\mu_H(t)-\gamma_2(t)-d_H(t)]\, H(t)+\alpha_1(t)B(t)+\gamma_1(t)F(t)+u(t)\\
B(t+1) &= [1-\alpha_1(t)-\beta_2(t)-d_B(t)]\, B(t)+\mu_2(t)T(t)\\
F(t+1) &= [1-\gamma_1(t)-d_F(t)]F(t)+\gamma_2(t)H(t)+\beta_2(t)B(t)\\
T(t+1) &= [1-\mu_2(t)-d_T(t)]T(t)+\mu_1(t)H(t)
\end{aligned}
\tag{5}
$$

Where $u(t)$is a control of the above system of difference equations.

4. Controllability of the System

In this section, we analyze the controllability of the system (5). We can write the system (5) as follows:

$$
x(t+1) = Ax(t)+Bu(t) \tag{6}
$$

Where $x(t)=[H(t),B(t),F(t),T(t)]^T$, A is a 4×4 matrix and B is a 4×1 matrix. $u(t)$ is a 1×1 vector.

$$
A=\begin{bmatrix}
1+\mu_H(t)-\gamma_2(t)-d_H(t) & \alpha_1(t) & \gamma_1(t) & 0\\
0 & 1-\alpha_1(t)-\beta_2(t)-d_B(t) & 0 & \mu_2(t)\\
\gamma_2(t) & \beta_2(t) & 1-\gamma_1(t)-d_F(t) & 0\\
\mu_1(t) & 0 & 0 & 1-\mu_2(t)-d_T(t)
\end{bmatrix} \tag{7}
$$

$$
B=\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \tag{8}
$$

Theorem 4.1. *System* (6) *is completely controllable if and only if rank* $W = 4$. *i.e) the following conditions are satisfied:*

$$\begin{aligned} &aei + i(de + ej) \neq 0 \\ &(af + hf)a + (cf + h^2)f + egi \neq 0 \\ &(ai + ji)a + cfi + ij^2 \neq 0 \end{aligned} \tag{9}$$

Proof. Let us consider the controllability matrix

$$W = [B, AB, A^2B, A^3B]$$

We know that

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 + \mu_H(t) - \gamma_2(t) - d_H(t) & \alpha_1(t) & \gamma_1(t) & 0 \\ 0 & 1 - \alpha_1(t) - \beta_2(t) - d_B(t) & 0 & \mu_2(t) \\ \gamma_2(t) & \beta_2(t) & 1 - \gamma_1(t) - d_F(t) & 0 \\ \mu_1(t) & 0 & 0 & 1 - \mu_2(t) - d_T(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \mu_H(t) - \gamma_2(t) - d_H(t) \\ 0 \\ \gamma_2(t) \\ \mu_1(t) \end{bmatrix} \tag{10}$$

Let us consider that

$$A = \begin{bmatrix} 1 + \mu_H(t) - \gamma_2(t) - d_H(t) & \alpha_1(t) & \gamma_1(t) & 0 \\ 0 & 1 - \alpha_1(t) - \beta_2(t) - d_B(t) & 0 & \mu_2(t) \\ \gamma_2(t) & \beta_2(t) & 1 - \gamma_1(t) - d_F(t) & 0 \\ \mu_1(t) & 0 & 0 & 1 - \mu_2(t) - d_T(t) \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c & 0 \\ 0 & d & 0 & e \\ f & g & h & 0 \\ i & 0 & 0 & j \end{bmatrix}$$

Where

$$a = 1 + \mu_H(t) - \gamma_2(t) - d_H(t) \qquad f = \gamma_2(t)$$
$$b = \alpha_1(t) \qquad g = \beta_2(t)$$
$$c = \gamma_1(t) \qquad h = 1 - \gamma_1(t) - d_F(t)$$
$$d = 1 - \alpha_1(t) - \beta_2(t) - d_B(t) \qquad i = \mu_1(t)$$
$$e = \mu_2(t) \qquad j = 1 - \mu_2(t) - d_T(t)$$

$$A^2B = \begin{bmatrix} a^2 + bf \\ ei \\ af + hf \\ ai + ji \end{bmatrix} \tag{11}$$

$$A^3B = \begin{bmatrix} a^3 + abf + (ac + ch)f + bei \\ aei + i(de + ej) \\ (af + hf)a + (cf + h^2)f + egi \\ (ai + ji)a + cfi + ij^2 \end{bmatrix} \tag{12}$$

Therefore we get the controllability matrix

$$W = \begin{bmatrix} 1 & a & a^2 + bf & a^3 + abf + (ac + ch)f + bei \\ 0 & 0 & ei & aei + i(de + ej) \\ 0 & f & af + hf & (af + hf)a + (cf + h^2)f + egi \\ 0 & i & ai + ji & (ai + ji)a + cfi + ij^2 \end{bmatrix} \tag{13}$$

The above matrix has rank 4 if and only if

$$aei + i(de + ej) \neq 0$$
$$(af + hf)a + (cf + h^2)f + egi \neq 0$$
$$(ai + ji)a + cfi + ij^2 \neq 0$$

□

Theorem 4.2. *The system* (6) *is controllable to the origin if and only if* $u(0) = -[ax_{01} + bx_{02} + cx_{03}]$ *and* $x_{04} = 0$.

Proof. Consider

$$x(0) = x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \\ x_{04} \end{bmatrix}$$

We know that, when $t = 0$, we get

$$x(1) = Ax_0 + Bu(0)$$

$$Ax_0 = = \begin{bmatrix} a & b & c & 0 \\ 0 & d & 0 & e \\ f & g & h & 0 \\ i & 0 & 0 & j \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \\ x_{04} \end{bmatrix}$$

Substituting we get

$$x(1) = \begin{bmatrix} ax_{01} + bx_{02} + cx_{03} \\ dx_{02} + ex_{04} \\ fx_{01} + gx_{02} + hx_{03} \\ ix_{01} + jx_{04} \end{bmatrix} + \begin{bmatrix} u(0) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The system (6) is controllable to the origin if and only if

$$u(0) = -[ax_{01} + bx_{02} + cx_{03}], \quad dx_{02} + ex_{04} = 0$$
$$fx_{01} + gx_{02} + hx_{03} = 0, \qquad ix_{01} + jx_{04} = 0$$

We obtain the above conditions if we consider $x_{04} = 0$.
i.e) if $x_{04} = 0$, we get $x_{01} = x_{02} = x_{03} = 0$ for all non zero coefficients.
Then we will have $x(1) = 0$, which implies that the system (6) is controllable to the origin. □

5. Conclusion

In this paper, we have constructed a controlled system of difference equations for a model on deforestation due to human population. Controllability is concerned with whether one can design control input to steer the state to arbitrarily values. We have analyzed the controllability conditions for the system using controllability matrix and we can also see that the system is controllable to the origin if $x_{04} = 0$.

References

[1] Hulin Wu and Wai-Yuan Tan, Modelling the HIV Epidemic: A State-Space Approach, Mathematical and Computer Modelling 32 (2000) 197-215.

[2] W.Y. Tan, The Chain Multinomial models of the HIV epidemiology in homosexual populations, Mathl. Comput. Modelling Vol.18, No.6, pp 29-72, (1993).

[3] Dubey, B., Sharma, S., Sinha, P., and Shukla.J.B :Modelling the depletion of forestry resources by population and population pressure augmented industrialization, Applied Mathematical Sciences, 33, 3002-3014, (2009).

[4] Manju Agarwal and Rachana Pathak: Conservation of Forestry Biomass and wildlife population: A Mathematical Model, Asian Journal of Mathematics and Computer Research, 4(1): 1-15, (2015).

[5] Saber N.Elaydi, An Introduction To Difference Equations, Second Edition, Springer, 1999.

[6] Elizabeth S and Preethi Victor, A Mathematical Model on deforestation due to human population and its effect on farm fields: Role of technology in its conservation, Smart Computing and Informatics Proceedings of the First International Conference on SCI 2016, Volume 2, Series ISSN:2190-3018, DOI:10.1007/978-981-10-5547-8, Series Vol:78, pp.425-432.

[7] Elizabeth S and Preethi Victor, Model on Deforestation due to Human Population and Role of Technology in its Conservation: A State Space Approach, International Journal of Difference Equations, Volume 12, Number 1, pp. 31-39, ISSN 0973-6069, (2017).