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Interval Oscillation Criteria for Self-adjoint Alpha-Fractional Matrix Differential Systems with Damping

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Abstract. In this paper, we are concerned with the oscillation criteria for self-adjoint alpha-fractional matrix differential system with damping term. By using the generalized Riccati technique and the averaging technique, some new oscillation criteria are obtained.

Key words: Interval criteria, Oscillation, α - Fractional derivative, Matrix differential systems.

Mathematics Subject classification 2010: 34A08, 34C10, 34K11.

1. Introduction

Fractional calculus is a branch of mathematics, which is as old as calculus but the applications are rather recent. It deals with differentiation and integration of arbitrary orders. It merges and generalizes the ideas of integer-order differentiation and n-fold integration whereas the fractional order models capture phenomena and properties that integer order neglect. The fractional order differential equations have been used to model several physical phenomena emerging in various Physical sciences, Biological, Ecological, Economics and Financial mathematics. See, for example [1,8,10,11,14,20,21,28-30,35] and the references cited therein.

The R-L and Caputo fractional derivatives are based on integral expressions and gamma functions which are nonlocal. In 2014, Khalil et al [19], introduced a new fractional derivative called the conformable derivative, using a limit definition

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analogous to that of standard derivative. The conformable derivative of Khalil was soon generalized by Katugampola which is reffered as katugampola fractional derivative or α-fractional derivative. See [2-4,7,17,18] and the references cited therein.

For the past few decades, the problem of oscillation and nonoscillation of solutions of matrix differential equations is one of the active area of research in the qualitative theory of matrix differential equations. See [5,6,9,12,13,15,16, 22-27,31-34].

An important tool in the study of oscillatory behavior of solutions for the matrix systems and corresponding the scalar analogue is the averaging technique which goes back as far as the classical properties of Wintner [32] and Hartman [13] giving sufficient oscillation conditions for those equations. The results of Wintner was improved by Kamenev [16], and further extensions of Kamenev's criterion have been obtained by Philos [27] and for the corresponding matrix system by Erbe, Kong and Ruan [12], Meng, Wang and Zhang [25], Kumari and Umamaheswaram [23] and Wang [31].

To the best of the our knowledge, there exists no literature and the oscillation of α- fractional matrix differential systems. Motivated by this gap, we proposed to initiate the following α- fractional matrix differential system of the form

$$D^{\alpha}\left(A(t)D^{\alpha}X(t)\right)+r(t)A(t)D^{\alpha}X(t)+B(t)X(t)=0,\ t\geq t_0, \tag{1}$$

where $A(t), B(t), X(t)$ are $n\times n$ real continuous matrix functions with $A(t), B(t)$ symmetric and $A(t)$ positive definite for $[t_0,\infty)$ $(A(t)>0, t\geq t_0)$, $r(t)\in C(I=[t_0,\infty),\mathbb{R}^+)$.

A solution of the system (1) is said to be *nontrivial* if det $X(t)\neq 0$ for atleast one $t\in[t_0,\infty)$, and a nontrivial solution $X(t)$ of (1) is said to be *prepared* or *self-conjugate* if

$$X^*(t)A(t)D^{\alpha}X(t)-D^{\alpha}X^*(t)A(t)X(t)=0,\ t\geq t_0,$$

where for any matrix A, the transpose of A is denoted by A^*. A prepared solution $X(t)$ of the system (1) is called *oscillatory* on $[t_0, \infty)$ if its determinant vanishes somewhere in $[T, \infty)$ for each $T \geq t_0$, otherwise, it is called *nonoscillatory*. Finally, the system (1) is called oscillatory on $[t_0, \infty)$ if every prepared solution is oscillatory.

However, from the Sturm separation Theorem, we see that oscillation is only an interval property, (i.e) if there exists a sequence of subintervals $[a_i, b_i]$ of $[t_0, \infty)$, as $a_i \to \infty$, such that for each i, there exists a solution of equation (1) that has atleast two zeros in $[a_i, b_i]$, then every solution of equation (1) is oscillatory, no matter how "bad" equation (1) is on the remaining parts of $[t_0, \infty)$.

In this paper, by using generalized Riccati technique and the averaging technique and by considering the function $H(t,s)k(s)$ which may not have a nonpositive partial derivative on $D_0 = \{(t,s) : t > s \geq t_0\}$ with respect to the second variable, we obtain some new general oscillation criteria for the system (1), that is, criteria given by the behavior of (1) (or of $A(t)$ and $B(t)$) only on a sequence of subintervals of $[t_0, \infty)$. By choosing appropriate functions H, k, ρ, we present a series of explicit oscillation criteria.

Hereafter we denote the trace of $n \times n$ matrix A by $tr(A)$. Further, E_n is the $n \times n$ identity matrix, and the eigenvalues of the $n \times n$ symmetric matrix A (an increasing order) are

$\underset{\min}{\lambda}[A] = \lambda_n[A] \leq \cdots \leq \underset{\max}{\lambda}[A]$.

Define $D_0 = \{(t,s) : t > s \geq t_0\}$, $D = \{(t,s) : t \geq s \geq t_0\}$.

2. Preliminaries

In this section, we give some basic definitions of the katugampola α- fractional derivatives, integrals and lemmas which are useful throughout this paper.

Definition 2.1. *[17] Let $y : [0, \infty) \to \mathbb{R}$ and $t > 0$. Then the fractional derivative of y of order α is given by*

$$D^\alpha y(t) := \lim_{\epsilon \to 0} \frac{y(te^{\epsilon t^{-\alpha}}) - y(t)}{\epsilon} \quad for \quad t > 0,$$

$\alpha \in (0,1]$. *If* y *is* α*- differentiable in some* $(0,a), a > 0$, *and* $\lim_{t\to 0^+} D^\alpha y(t)$ *exists, then we define*

$$D^\alpha y(0) := \lim_{t\to 0^+} D^\alpha y(t).$$

The α-fractional derivative satisfies the following properties.

Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t > 0$. Then

(p_1) $D^\alpha(t^n) = nt^{n-\alpha}$ for all $n \in \mathbb{R}$.

(p_2) $D^\alpha(c) = 0$ for all constant functions, $f(t) = c$.

(p_3) $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$.

(p_4) $D^\alpha\left(\dfrac{f}{g}\right) = \dfrac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$.

(p_5) $D^\alpha(f \circ g)(t) = D^\alpha f(g(t))D^\alpha(g)(t)$.

(p_6) If f is differentiable, then $D^\alpha(f)(t) = t^{1-\alpha}\dfrac{df(t)}{dt}$.

Definition 2.2. *[17] Let* $a \geq 0$ *and* $t \geq a$. *Also, let* y *be a function defined on* $(a,t]$ *and* $\alpha \in \mathbb{R}$. *Then, the* α*- fractional integral of* y *is given by*

$$I_a^\alpha y(t) := \int_a^t \frac{y(x)}{x^{1-\alpha}} dx$$

if the Riemann improper integral exists.

Remark 1. *Throughout the paper, we use the following notation. Further, if* $X(t) = (X_{i,j}(t))_{n\times n}$ *then*

$$D^\alpha X(t) = (D^\alpha X_{i,j}(t))_{n\times n}$$

where

$$D^\alpha X_{i,j}(t) := \lim_{\epsilon\to 0} \frac{X_{i,j}(te^{\epsilon t^{-\alpha}}) - X_{i,j}(t)}{\epsilon}.$$

Also, if each $X_{i,j}(t)$ *is differentiable, then*

$$D^\alpha X_{i,j}(t) = t^{1-\alpha} X'_{i,j}(t)$$

and hence

$$D^\alpha X(t) = t^{1-\alpha} X'(t).$$

3. Main Results

In this section, we study oscillatory behavior of solutions of the α-fractional matrix differential system (1).

Theorem 3.1. *Suppose that the functions $H \in C(D, \mathbb{R}), h_1, h_2 \in C(D_0, \mathbb{R})$ and $k, \rho \in C^\alpha([t_0, \infty), (0, \infty))$ satisfy the following conditions:*

(H_1) $H(t,t) = 0$ *for* $t \geq t_0, H(t,s) > 0$ *on* D_0;

(H_2) $\dfrac{\partial(H(t,s)k(t))}{\partial t} - \left(r(t)t^{\alpha-1} - \dfrac{\rho'(t)}{\rho(t)}\right) H(t,s)k(t) = h_1(t,s)$ *for all* $(t,s) \in D_0$;

(H_3) $\dfrac{\partial(H(t,s)k(s))}{\partial s} - \left(r(s)s^{\alpha-1} - \dfrac{\rho'(s)}{\rho(s)}\right) H(t,s)k(s) = -h_2(t,s)$ *for all* $(t,s) \in D_0$.

Assume also that for each sufficiently large $T_0 \geq t_0$, there exist $a, b, c \in \mathbb{R}$ with $T_0 \leq a < c < b$ such that

$$\frac{1}{H(c,a)}\lambda_n \left[\int_a^c \left\{H(s,a)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}\frac{h_1^2(s,a)A(s)\rho(s)}{H(s,a)k(s)s^{\alpha-1}}\right\} ds\right] +$$

$$\frac{1}{H(b,c)}\lambda_1 \left[\int_c^b \left\{H(b,s)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}\frac{h_2^2(b,s)A(s)\rho(s)}{H(b,s)k(s)s^{\alpha-1}}\right\} ds\right] > 0. \qquad (2)$$

Then the system (1) is oscillatory.

Proof. Assume that there exists a prepared solution $X(t)$ of the system (1) which is not oscillatory. Without loss of generality, we may assume that det $X(t) \neq 0$ for $t \geq t_0$. Define

$$W(t) = \rho(t)A(t)D^\alpha X(t)X^{-1}(t) \ \ for\ t \geq t_0. \qquad (3)$$

By α-differentiating the matrix (3) and making use of (1) we find that $W(t)$ satisfies Riccati equation for $t \in [t_0, \infty)$;

$$D^\alpha W(t) = \frac{D^\alpha \rho(t)}{\rho(t)}W(t) - r(t)W(t) - \rho(t)B(t) - \frac{1}{\rho(t)}W(t)A^{-1}(t)W(t).$$

Multiplying by $t^{\alpha-1}$ on both sides and apply (p_6), we get

$$t^{\alpha-1}\rho(t)B(t) = -W'(t) - \left(r(t)t^{\alpha-1} - \frac{\rho'(t)}{\rho(t)}\right)W(t) - t^{\alpha-1}\frac{1}{\rho(t)}W(t)A^{-1}(t)W(t). \quad (4)$$

On multiplying (4) by $H(t,s)k(s)$ and integrating with respect to s from c to t for $t \in [c, b)$, we obtain

$$\int_c^t H(t,s)k(s)s^{\alpha-1}\rho(s)B(s)ds$$
$$= -\int_c^t H(t,s)k(s)W'(s)ds - \int_c^t H(t,s)k(s)\left(r(s)s^{\alpha-1} - \frac{\rho'(s)}{\rho(s)}\right)W(s)ds$$
$$-\int_c^t H(t,s)k(s)s^{\alpha-1}\frac{1}{\rho(s)}W(s)A^{-1}(s)W(s)ds.$$

$$= H(t,c)k(c)W(c) - \int_c^t \left[-\frac{\partial(H(t,s)k(s))}{\partial s} + \left(r(s)s^{\alpha-1} - \frac{\rho'(s)}{\rho(s)}\right)H(t,s)k(s)\right]W(s)ds$$
$$-\int_c^t H(t,s)k(s)s^{\alpha-1}\frac{1}{\rho(s)}W(s)A^{-1}(s)W(s)ds.$$
$$= H(t,c)k(c)W(c) - \int_c^t h_2(t,s)W(s)ds - \int_c^t H(t,s)k(s)s^{\alpha-1}\frac{1}{\rho(s)}W(s)A^{-1}(s)W(s)ds.$$

Since $A(t) > 0$, we can let

$$V(t) = \left[\frac{1}{\rho(t)}A^{-1}(t)\right]^{\frac{1}{2}}.$$

Substituting $V(t)$ into the above equation, we obtain

$$\int_c^t H(t,s)k(s)s^{\alpha-1}\rho(s)B(s)ds$$
$$= H(t,c)k(c)W(c) - \int_c^t h_2(t,s)V^{-1}(s)V(s)W(s)V^{-1}(s)V(s)ds$$
$$-\int_c^t H(t,s)k(s)s^{\alpha-1}V^{-1}(s)\left[V(s)W(s)V(s)\right]\left[V(s)W(s)V(s)\right]V^{-1}(s)ds.$$
$$= H(t,c)k(c)W(c) + \frac{1}{4}\int_c^t \frac{h_2^2(t,s)A(s)\rho(s)}{H(t,s)k(s)s^{\alpha-1}}ds$$

$$-\int_c^t V^{-1}(s)\left\{|H(t,s)k(s)s^{\alpha-1}|^{\frac{1}{2}}\left[V(s)W(s)V(s)\right] + \frac{1}{2}\frac{h_2(t,s)E_n}{|H(t,s)k(s)s^{\alpha-1}|^{\frac{1}{2}}}\right\}^2 V^{-1}(s)ds.$$

Thus we get

$$\int_c^t \left\{H(t,s)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}\frac{h_2^2(t,s)A(s)\rho(s)}{H(t,s)k(s)s^{\alpha-1}}\right\}ds \le H(t,c)k(c)W(c).$$

Thus

$$\lambda_1\left[\int_c^t\left\{H(t,s)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}\frac{h_2^2(t,s)A(s)\rho(s)}{H(t,s)k(s)s^{\alpha-1}}\right\}ds\right] \le \lambda_1\left[H(t,c)k(c)W(c)\right]. \quad (5)$$

Letting $t \to b^-$ in (5) and dividing both sides by $H(b,c)$, we obtain

$$\frac{1}{H(b,c)}\lambda_1\left[\int_c^b\left\{H(b,s)k(s)s^{\alpha-1}\rho(s)B(s)-\frac{1}{4}\frac{h_2^2(b,s)A(s)\rho(s)}{H(b,s)k(s)s^{\alpha-1}}\right\}ds\right]\leq\lambda_1\left[k(c)W(c)\right]. \quad (6)$$

Similarly to the above proof, multiplying (4), with t replaced by s, by $H(s,t)k(s)$ and integrating with respect to s from t to c for $t \in (a,c]$, we obtain

$$\begin{aligned}\int_t^c & H(s,t)k(s)s^{\alpha-1}\rho(s)B(s)ds\\ &= -\int_t^c H(s,t)k(s)W'(s)ds - \int_t^c H(s,t)k(s)\left(r(s)s^{\alpha-1}-\frac{\rho'(s)}{\rho(s)}\right)W(s)ds\\ &\quad -\int_t^c H(s,t)k(s)s^{\alpha-1}\frac{1}{\rho(s)}W(s)A^{-1}(s)W(s)ds.\end{aligned}$$

$$\begin{aligned}&= -H(c,t)k(c)W(c) - \int_t^c\left[-\frac{\partial(H(s,t)k(s))}{\partial s}+\left(r(s)s^{\alpha-1}-\frac{\rho'(s)}{\rho(s)}\right)H(s,t)k(s)\right]W(s)ds\\ &\quad -\int_t^c H(s,t)k(s)s^{\alpha-1}\frac{1}{\rho(s)}W(s)A^{-1}(s)W(s)ds.\\ &= -H(c,t)k(c)W(c) - \int_t^c h_1(s,t)W(s)ds - \int_t^c H(s,t)k(s)s^{\alpha-1}\frac{1}{\rho(s)}W(s)A^{-1}(s)W(s)ds.\end{aligned}$$

Since $A(t) > 0$, we can again let

$$V(t)=\left[\frac{1}{\rho(t)}A^{-1}(t)\right]^{\frac{1}{2}}.$$

Substituting $V(t)$ into the above equation, we get

$$\begin{aligned}\int_t^c & H(s,t)k(s)s^{\alpha-1}\rho(s)B(s)ds\\ &= -H(c,t)k(c)W(c) + \int_t^c h_1(s,t)V^{-1}(s)V(s)W(s)V^{-1}(s)V(s)ds\\ &\quad -\int_t^c H(s,t)k(s)s^{\alpha-1}V^{-1}(s)\left[V(s)W(s)V(s)\right]\left[V(s)W(s)V(s)\right]V^{-1}(s)ds.\\ &= -H(c,t)k(c)W(c) + \frac{1}{4}\int_t^c \frac{h_1^2(s,t)A(s)\rho(s)}{H(s,t)k(s)s^{\alpha-1}}ds\\ &\quad -\int_t^c V^{-1}(s)\left\{|H(s,t)k(s)s^{\alpha-1}|^{\frac{1}{2}}\left[V(s)W(s)V(s)\right]+\frac{1}{2}\frac{h_1(s,t)E_n}{|H(t,s)k(s)s^{\alpha-1}|^{\frac{1}{2}}}\right\}^2 V^{-1}(s)ds.\end{aligned}$$

Thus we obtain

$$\int_t^c \left\{ H(s,t)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}\frac{h_1^2(s,t)A(s)\rho(s)}{H(s,t)k(s)s^{\alpha-1}} \right\} ds \leq -H(c,t)k(c)W(c).$$

Thus

$$\begin{aligned}\lambda_n \left[\int_t^c \left\{ H(s,t)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}\frac{h_1^2(s,t)A(s)\rho(s)}{H(s,t)k(s)s^{\alpha-1}} \right\} ds \right] &\leq -\lambda_n \left[H(t,c)k(c)W(c) \right] \\ &= -\lambda_1 \left[H(t,c)k(c)W(c) \right],\end{aligned}$$

where $t \in (a,c]$. Letting $t \to a^+$ in the above inequality and dividing both sides by $H(c,a)$, we obtain

$$\frac{1}{H(c,a)}\lambda_n \left[\int_a^c \left\{ H(s,a)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}\frac{h_1^2(s,a)A(s)\rho(s)}{H(s,a)k(s)s^{\alpha-1}} \right\} ds \right] \leq -\lambda_1 \left[k(c)W(c) \right]. \quad (7)$$

Now we claim that $\det X(t)$, where $x(t)$ is any prepared solution of (1), has atleast one zero in (a,b). Assume to the contrary. Adding (6) and (7), we have an inequality which contradicts the assumption (2). Thus, conclusion holds. Pick up a sequence $\{T_i\} \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. By the assumptions of Theorem 3.1, for each $i \in \mathbb{N}$, there exist $a_i, b_i, c_i \in \mathbb{R}$ such that $T_i \leq a_i < b_i < c_i$, and (2) holds with a, b, c replaced by a_i, b_i, c_i, respectively. From the above claim, the determinant of every prepared solution $X(t)$ has at least one zero $t_i \in (a_i, b_i)$. Noting that $t_i > a_i > T_i$, $i \in \mathbb{N}$, we see that $detX(t)$ has arbitrarily large zeros. Thus the system (1) is oscillatory. The proof is complete. □

Under a modification of the hypotheses of Theorem 3.1, we can obtain the following result.

Corollary 3.0.1. *Under the assumptions of Theorem 3.1 with the condition (2) replaced by*

$$\begin{aligned}&\frac{1}{H(c,a)}\int_a^c H(s,a)k(s)s^{\alpha-1}\rho(s)trB(s)ds + \frac{1}{H(b,c)}\int_c^b H(b,s)k(s)s^{\alpha-1}\rho(s)trB(s)ds \\ &> \frac{1}{4}\left[\frac{1}{H(c,a)}\int_a^c \frac{h_1^2(s,a)trA(s)\rho(s)}{H(s,a)k(s)s^{\alpha-1}}ds + \frac{1}{H(b,c)}\int_c^b \frac{h_2^2(b,s)trA(s)\rho(s)}{H(b,s)k(s)s^{\alpha-1}}ds \right], \quad (8)\end{aligned}$$

the system (1) is oscillatory.

This proof is similar to that of Theorem 3.1.

Theorem 3.2. *Let the assumptions of Theorem 3.1 with (2) replaced by*

$$\limsup_{t\to\infty} \lambda_n \left[\int_l^t \left\{H(s,l)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}\frac{h_1^2(s,l)A(s)\rho(s)}{H(s,l)k(s)s^{\alpha-1}}\right\} ds\right] > 0 \text{ and} \quad (9)$$

$$\limsup_{t\to\infty} \lambda_1 \left[\int_l^t \left\{H(t,s)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}\frac{h_2^2(t,s)A(s)\rho(s)}{H(t,s)k(s)s^{\alpha-1}}\right\} ds\right] > 0 \quad (10)$$

for each $l \geq t_0$. Then the system (1) is oscillatory.

Corollary 3.0.2. *Let the assumptions of Theorem 3.1 with the condition including (2) replaced by*

$$\limsup_{t\to\infty} \int_l^t \left\{H(s,l)k(s)s^{\alpha-1}\rho(s)trB(s) - \frac{1}{4}\frac{h_1^2(s,l)trA(s)\rho(s)}{H(s,l)k(s)s^{\alpha-1}}\right\} ds > 0 \text{ and}$$

$$\limsup_{t\to\infty} \int_l^t \left\{H(t,s)k(s)s^{\alpha-1}\rho(s)trB(s) - \frac{1}{4}\frac{h_2^2(t,s)trA(s)\rho(s)}{H(t,s)k(s)s^{\alpha-1}}\right\} ds > 0$$

for each $l \geq t_0$, the system (1) is oscillatory.

If in Theorem 3.1 and Theorem 3.2, Corollary 3.1 and Corollary 3.2, $h_1(t,s)$ and $h_2(t,s)$ replaced by $h_1(t,s)\sqrt{H(t,s)k(s)s^{\alpha-1}}$ and $h_2(t,s)\sqrt{H(t,s)k(s)s^{\alpha-1}}$ respectively, we can obtain the following results. The proofs are similar.

Theorem 3.3. *Assume $H \in C(D,\mathbb{R})$ satisfy the condition (H_1) in Theorem 3.1. Suppose that there exist $h_1, h_2 \in C(D_0,\mathbb{R})$ and $k, \rho \in C^\alpha([t_0,\infty),(0,\infty))$ such that*

(H_2) $$\frac{\partial(H(t,s)k(t))}{\partial t} - \left(r(t)t^{\alpha-1} - \frac{\rho'(t)}{\rho(t)}\right)H(t,s)k(t) = h_1(t,s)\sqrt{H(t,s)k(t)t^{\alpha-1}}$$ *for all $(t,s) \in D_0$;*

(H_3) $$\frac{\partial(H(t,s)k(s))}{\partial s} - \left(r(s)s^{\alpha-1} - \frac{\rho'(s)}{\rho(s)}\right)H(t,s)k(s) = -h_2(t,s)\sqrt{H(t,s)k(s)s^{\alpha-1}}$$ *for all $(t,s) \in D_0$.*

Assume also that for each sufficiently large $T_0 \geq t_0$, there exist $a,b,c \in \mathbb{R}$ with $T_0 \leq a < c < b$ such that

$$\frac{1}{H(c,a)}\lambda_n \left[\int_a^c \left\{H(s,a)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}h_1^2(s,a)A(s)\rho(s)\right\} ds\right] +$$
$$\frac{1}{H(b,c)}\lambda_1 \left[\int_c^b \left\{H(b,s)k(s)s^{\alpha-1}\rho(s)B(s) - \frac{1}{4}h_2^2(b,s)A(s)\rho(s)\right\} ds\right] > 0. \quad (11)$$

Then the system (1) is oscillatory.

Corollary 3.0.3. *Let the assumptions of Theorem 3.3 with the condition (11) replaced by*

$$\frac{1}{H(c,a)}\int_a^c H(s,a)k(s)s^{\alpha-1}\rho(s)trB(s)ds$$
$$+\frac{1}{H(b,c)}\int_c^b H(b,s)k(s)s^{\alpha-1}\rho(s)trB(s)ds$$
$$>\frac{1}{4}\left[\frac{1}{H(c,a)}\int_a^c h_1^2(s,a)trA(s)\rho(s)ds+\frac{1}{H(b,c)}\int_c^b h_2^2(b,s)trA(s)\rho(s)ds\right], \quad (12)$$

the system (1) is oscillatory.

Theorem 3.4. *Under the assumptions of Theorem 3.3 with the condition (11) replaced by*

$$\limsup_{t\to\infty}\lambda_n\left[\int_l^t\left\{H(s,l)k(s)s^{\alpha-1}\rho(s)B(s)-\frac{1}{4}h_1^2(s,l)A(s)\rho(s)\right\}ds\right]>0 \textit{ and} \quad (13)$$

$$\limsup_{t\to\infty}\lambda_1\left[\int_l^t\left\{H(t,s)k(s)s^{\alpha-1}\rho(s)B(s)-\frac{1}{4}h_2^2(t,s)A(s)\rho(s)\right\}ds\right]>0 \quad (14)$$

for each $l\geq t_0$. Then the system (1) is oscillatory.

Corollary 3.0.4. *Let the assumptions of Theorem 3.3 with the condition including (11) replaced by*

$$\limsup_{t\to\infty}\int_l^t\left\{H(s,l)k(s)s^{\alpha-1}\rho(s)trB(s)-\frac{1}{4}h_1^2(s,l)trA(s)\rho(s)\right\}ds>0 \textit{ and}$$
$$\limsup_{t\to\infty}\int_l^t\left\{H(t,s)k(s)s^{\alpha-1}\rho(s)trB(s)-\frac{1}{4}h_2^2(t,s)trA(s)\rho(s)\right\}ds>0$$

for each $l\geq t_0$, the system (1) is oscillatory.

4. Conclusion

In this paper, we have established some oscillation results for α- fractional matrix differential system using Riccati transformation and averaging technique. Our results are essentially new, have improved and generalized some of the results.

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