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Some Integral Theorems Based on Generalized Difference Operator and its Equation

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Abstract. In this paper, we establish convolution theorem, discrete Fourier transform, discrete Green's and Gauss Divergence theorems using a generalized difference operator and its equation. Also we present few examples verified by MATLAB to illustrate the theorems.

Key words: Generalized difference equation, Convolution Theorem, Green's Theorem, Divergence theorem.

AMS Subject classification: 39A70, 47B39, 39A10, 49M.

1. Introduction

Fractional calculus and fractional difference equations have undergone expanded study in recent years as a considerable interest both in Mathematics and in applications. They were applied in modeling of many physical and chemical processes and in engineering [1–4]. The theory of generalized difference operator Δ_ℓ defined as $\Delta_\ell v(k) = v(k+\ell) - v(k)$ is developed in [7]. So in this paper, we extend the theory of Δ_ℓ to the calculus of real functions for finding the values of some integral theorems using the inverse of generalized difference operator Δ_ℓ.

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We complete this introduction with a brief description of the paper. This paper has five sections. In Section 2, we present some preliminary results. In Section 3, we derive convolution theorem for discrete Fourier transform for the sequence of real numbers. In section 4, we present the discrete Green's theorem and Gauss Divergent theorem.

2. Preliminaries

In this section, we present some notations and preliminary results on generalized difference operator which will be used for the subsequent discussions. For simplicity, we use the following notations:

(*i*) $\underset{\ell_{1\to n}}{\Delta^{-1}} = \Delta_{\ell_1}^{-1}\Delta_{\ell_2}^{-1}\Delta_{\ell_3}^{-1}\cdots\Delta_{\ell_n}^{-1}$; (*ii*) $\underset{\ell_{1,n}}{\Delta^{-1}} = \Delta_{\ell_1}^{-1}\Delta_{\ell_n}^{-1}$

Lemma 2.1. *[7] Let s_r^m and S_r^m be the Stirling numbers of first and second kinds respectively and $k_\ell^{(m)} = k(k-\ell)\cdots(k-(m-1)\ell)$. Then we have*

$$k_q^{(m)} = \sum_{r=1}^{m} s_r^m q^{m-r} k^r, \qquad k^m = \sum_{r=1}^{m} S_r^m q^{m-r} k_q^{(r)} \tag{1}$$

and

$$\Delta_\ell^{-n} k_\ell^{(m)} = \frac{k_\ell^{(m+n)}}{\ell^n (m+n)^{(n)}}. \tag{2}$$

3. Discrete Convolution Theorem and Fourier Transform

In this section, we establish discrete convolution theorem and discrete Fourier transform based on generalized operator Δ_ℓ. When $\ell \to 0$, we get convolution theorem and Fourier transform.

Definition 3.1. *Let f and g be two bounded functions on $(-\infty, +\infty)$. Then the convolution of f and g is defined as*

$$h_\ell(x) = \ell\Delta_\ell^{-1} f(t)g(x-t)\Big|_{-\infty}^{\infty}. \tag{3}$$

We also write $h = f * g$ to denote this function. It is easy to see that $f * g = g * f$.

Theorem 3.2. *Let $\mathbb{R} = (-\infty, \infty)$. Assume that $f, g \in L(\mathbb{R})$ and that either f or g is bounded on $\mathbb{R}$. Then the discrete convolution intgral*

$$h_\ell(x) = \ell\Delta_\ell^{-1} f(t)g(x-t)\Big|_{-\infty}^{\infty} \tag{4}$$

exists for every x in $\mathbb{R}$ and the function h so defined is bounded on $\mathbb{R}$. If, in addition, the bounded function f or g is continuous on $\mathbb{R}$, then h is also continuous on $\mathbb{R}$ and $h \in L(\mathbb{R})$.

Proof. Since $f * g = g * f$, it suffices to consider the case in which g is bounded. Suppose $|g| \leq M$. Then

$$|f(t)g(x-t)| \leq M\,|f(t)|\,. \tag{5}$$

and f(t)g(x-t) is a measurable function of t on $\mathbb{R}$. So the discrete integral for h(x) exists and from (3) and (5), h is bounded on $\mathbb{R}$. Also if g is continuous on $\mathbb{R}$, then h is continuous on $\mathbb{R}$. Now for every compact interval $[a,b]$, we have
$\left(\ell\Delta_\ell^{-1}\,|h_\ell(x)|\,\right)\Big|_a^b = \ell\Delta_\ell^{-1}\,|f(t)|_{-\infty}^{\infty}\,\ell\Delta_\ell^{-1}\,|g(x-t)|_a^b \leq \ell\Delta_\ell^{-1}\,|f(t)|_{-\infty}^{\infty}\,\ell\Delta_\ell^{-1}\,|g(y)|_{-\infty}^{\infty}\,.$
so $h \in L(\mathbb{R})$. ☐

Here, we present the Discrete Convolution and Fourier Transforms.

Theorem 3.3. *Let $L(\mathbb{R})$ =set of all Lebesgue integrable functions on $\mathbb{R}$. Assume that $f, g \in L(\mathbb{R})$ and that atleast one of f or g is continous and bounded on $\mathbb{R}$. Let h_ℓ denotes the convolution and $h_\ell = f * g$. Then, for every real u, we have*

$$\ell\Delta_\ell^{-1}h_\ell(x)e^{-ixu}\Big|_{-\infty}^{\infty} = \left(\ell\Delta_\ell^{-1}f(t)e^{-itu}\right)\Big|_{-\infty}^{\infty}\left(\ell\Delta_\ell^{-1}g(y)e^{-iyu}\right)\Big|_{-\infty}^{\infty}. \tag{6}$$

Proof. Assume that g is continuous and bounded on R. Let a_n and b_n be two increasing sequences of positive real numbers such that $a_n \to +\infty$ and $b_n \to +\infty$. Define a sequence of function f_n on $\mathbb{R}$ as $f_n(t) = \ell\Delta_\ell^{-1}e^{-iux}g(x-t)\Big|_{-a_n}^{b_n}$.
Since $\left|\ell\Delta_\ell^{-1}e^{-iux}g(x-t)\right|\Big|_{-a_n}^{b_n} \leq |g|_{-\infty}^{\infty}$ for all compact intervals [a,b],

$$\lim_{n\to\infty} f_n(t) = \ell\Delta_\ell^{-1}e^{-iux}g(x-t)\Big|_{-\infty}^{\infty} = \ell\Delta_\ell^{-1}e^{-iu(t+y)}g(y)\Big|_{-\infty}^{\infty}., \tag{7}$$

Now, continuity of f_n on $\mathbb{R}$ results $f.f_n$ is measurable on $\mathbb{R}$ and hence $f.f_n$ is Discrete Lebesgue-integral on $\mathbb{R}$. Also by Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\ell\Delta_\ell^{-1}f(t)f_n(t)\Big|_{-\infty}^{\infty} = \left(\ell\Delta_\ell^{-1}f(t)e^{-itu}\right)\Big|_{-\infty}^{\infty}\left(\ell\Delta_\ell^{-1}g(y)e^{-iyu}\right)\Big|_{-\infty}^{\infty}. \tag{8}$$

But from (7) and (3), $\lim_{n\to\infty}\ell\Delta_\ell^{-1}f(t)f_n(t)\Big|_{-\infty}^{\infty} = \ell\Delta_\ell^{-1}e^{-iux}h(x)\Big|_{-a_n}^{b_n}$, which completes the proof. The Discrete integral on the left also exists as an improper

Riemann integral is continuous and bounded on R and $\ell\Delta_\ell^{-1}\left|h(x)e^{-iux}\right| \le \Delta_\ell^{-1}h\Big|_{-\infty}^{\infty}$ for every compact interval [a,b]. $\square$

An example verified by MATLAB is given below to illustate Theorem 3.3:

Example 3.4. *consider the following functions*

$$f(t) = \begin{cases} t : t \in (0,\infty) \\ 0 : t \in (-\infty, 0)\,, \end{cases} \qquad g(y) = \begin{cases} e^{-y-2t} : t \in (0,\infty) \\ 0 : t \in (-\infty, 0) \end{cases}.$$

Then, we have $h_\ell(x) = \ell\Delta_\ell^{-1}f(t)g(x-t)\Big|_{-\infty}^{\infty} = \ell\Delta_\ell^{-1}te^{-(x+t)}\Big|_0^{\infty} = \dfrac{\ell^2 e^{-(x+\ell)}}{(e^{-\ell}-1)^2}$.

Also $\ell\Delta_\ell^{-1}h(x)e^{-ixu}\Big|_{-\infty}^{\infty} = \dfrac{\ell^3}{(e^{-\ell}-1)^2(e^{-\ell(1+iu)}-1)}\left\{e^{-b(1+iu)-\ell} - e^{a(1+iu)-\ell}\right\}$ and

$$(\ell\Delta_\ell^{-1}f(t)e^{-itu})\Big|_{-\infty}^{\infty}(\ell\Delta_\ell^{-1}g(y)e^{-iyu})\Big|_{-\infty}^{\infty} = (\ell\Delta_\ell^{-1}te^{-itu})\Big|_0^{\infty}\,(\ell\Delta_\ell^{-1}e^{-(x+t)}e^{-iu(x-t)})\Big|_{-a}^{b}$$

$$= \frac{\ell^3}{(e^{-\ell}-1)^2(e^{-\ell(1+iu)}-1)}\left\{e^{-b(1+iu)-\ell} - e^{a(1+iu)-\ell}\right\}. \tag{9}$$

Hence from (3) and (9), Theorem 3.3 is verified.

MATLAB Coding : $((exp(-5*(1+i*u)+\ell) - exp(5*(1+i*u)-\ell))*\ell.\wedge 3)./((exp(\ell)-1).\wedge 2*(exp(-\ell*(1+i*u))-1));$

The portrait of the given functions $f(t)$ and $g(y)$ with $y=-10$ before applying the inverse operator is given in fig1 and in fig2, we plot the result of convolving $f(t)$ with $g(y)$ after applying the inverse operator for different values of ℓ with $a=b=5$.

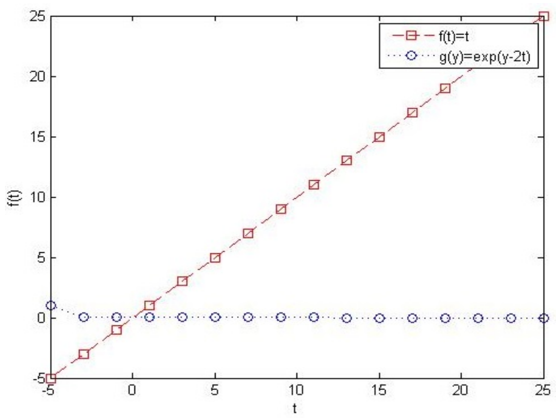


fig 1

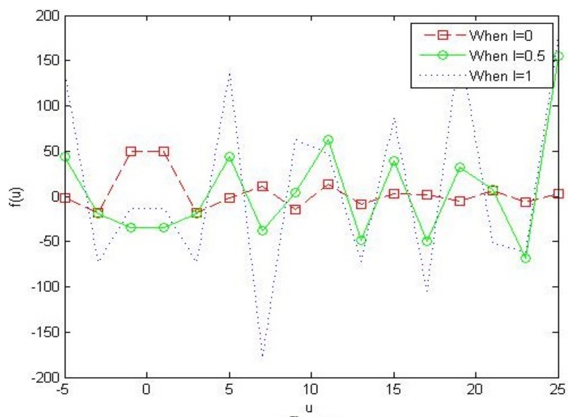


fig 2

Similarly, we can apply Δ_ℓ theory to Green's and Gauss divergence theorems.

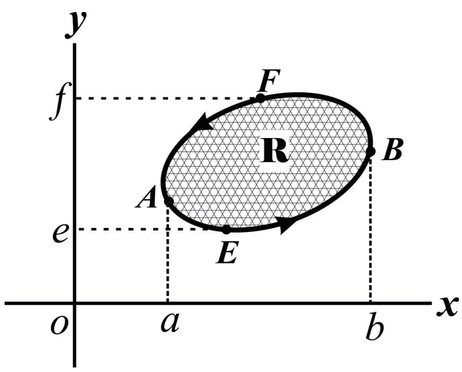
4. Discrete Green's and Gauss Divergent Theorems

In this section we derive the generalized discrete Green's Theorem and discrete Gauss divergence theorem using Δ_ℓ^{-1}.

Theorem 4.1. *If R is a closed region of the xy plane bounded by a simple closed curve C, which is traversed in the anticlockwise direction and M and N are functions of x and y, having continuous partial derivatives in R, then*

$$\big(\ell_1\Delta_{\ell_1}^{-1}M + \ell_2\Delta_{\ell_2}^{-1}N\big)\big|_C = \ell_1\ell_2 \underset{\ell_{1,2}}{\Delta^{-1}} \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\Big|_R. \tag{10}$$

Proof. Let the equations of the curves AEB and AFB be $y = Y_1(x)$ and $y = Y_2(x)$ respectively. If R is the region bounded by C, then we have



$$\ell_1\ell_2 \underset{\ell_{1,2}}{\Delta^{-1}} \frac{\partial M}{\partial y}\Big|_R = \ell_1\Delta_{\ell_1}^{-1}\big(M(x,Y_2) - M(x,Y_1)\big)\big|_a^b = -\ell_1\Delta_{\ell_1}^{-1}M\big|_C. \tag{11}$$

If $x = X_1(y)$ and $x = X_2(y)$ are the curves EAF and EBF respectively, then

$$\ell_2\Delta_{\ell_2}^{-1}N\big|_C = \ell_1\ell_2 \underset{\ell_{1,2}}{\Delta^{-1}} \frac{\partial N}{\partial x}\Big|_R. \tag{12}$$

Hence the proof follows from (11) and (12). $\square$

Example 4.2. *Let M=$3x^2 - 8y^2$, N=$4y - 6xy$ and C is the rectangle formed by the lines $x = 0, x = 1$, $y = 0, y = 2$ in the XOY plane.*
Then using (10) and from (2), we get

$$\big(\ell_1\Delta_{\ell_1}^{-1}M + \ell_2\Delta_{\ell_2}^{-1}N\big)\big|_C = \ell_1\ell_2 \underset{\ell_{1,2}}{\Delta^{-1}} 10y\Big|_0^2\Big|_0^1 = 10(2-\ell_2).$$

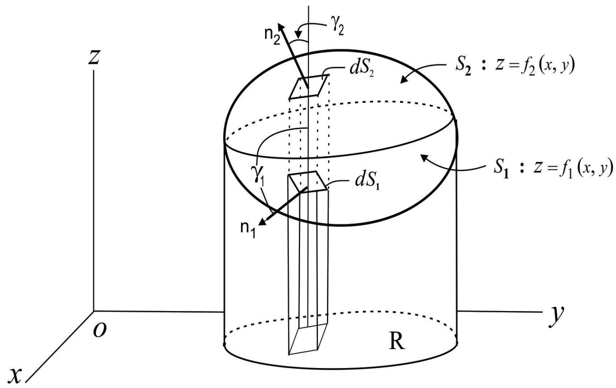
Remark 4.3. *When $\ell_2 \to 0$,* $\big(\ell_1\Delta_{\ell_1}^{-1}(3x^2 - 8y^2) + \ell_2\Delta_{\ell_2}^{-1}(4y - 6xy)\big)\big|_C = 20.$

The following is the discrete version of generalized Gauss Divergence Theorem.

Theorem 4.4. *If V is the volume of a closed surface S and A is a vector point function with continous first partial derivatives in V, then we have*

$$\Big(\ell_2\ell_3 \underset{\ell_{2,3}}{\Delta^{-1}} A_1 + \ell_1\ell_3 \underset{\ell_{1,3}}{\Delta^{-1}} A_2 + \ell_1\ell_2 \underset{\ell_{1,2}}{\Delta^{-1}} A_3\Big)\Big|_S = \ell_1\ell_2\ell_3 \underset{\ell_{1\to3}}{\Delta^{-1}}(\nabla.\vec{A})\Big|_V. \tag{13}$$

Proof. Let S be a closed surface such that no line parallel to the co-ordinate axes intersect it in more than two points. Let the cylinder whose generators are parallel to the z-axis and which envelops the surface S, touch S along the curve 'C' and intersects the xy plane along the curve. Now the curve C divides the surface S into two parts, say S_1 and S_2, whose equations are $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. If $A = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ and R is the area enclosed by C in the xy plane, then by considering the volume integral of $\frac{\partial A_3}{\partial z}$ over the region enclosed by S, we get



$$\ell_1\ell_2\ell_3 \underset{\ell_{1\to3}}{\Delta^{-1}} \frac{\partial A_3}{\partial z}\Big|_V = \ell_1\ell_2\ell_3 \underset{\ell_{1\to3}}{\Delta^{-1}} \frac{\partial A_3}{\partial z}\Big|_{z=f_1}^{z=f_2}\Big|_R = \ell_1\ell_2 \underset{\ell_{1,2}}{\Delta^{-1}} \big(A_3(x,y,f_2) - A_3(x,y,f_1)\big)\Big|_R$$

$$\ell_1\ell_2\ell_3 \underset{\ell_{1\to3}}{\Delta^{-1}} \frac{\partial A_3}{\partial z}\Big|_V = \ell_1\ell_2 \underset{\ell_{1,2}}{\Delta^{-1}} \big(A_3\vec{k}.\hat{n}\big)\Big|_S. \tag{14}$$

Similarly, by projecting S on the other co-ordinate planes, we get

$$\ell_1\ell_2\ell_3 \underset{\ell_{1\to3}}{\Delta^{-1}} \frac{\partial A_1}{\partial x}\Big|_V = \ell_2\ell_3 \underset{\ell_{2,3}}{\Delta^{-1}} \big(A_1\vec{i}.\hat{n}\big)\Big|_S, \tag{15}$$

and

$$\ell_1\ell_2\ell_3 \underset{\ell_{1\to3}}{\Delta^{-1}} \frac{\partial A_2}{\partial y}\Big|_V = \ell_1\ell_3 \underset{\ell_{1,3}}{\Delta^{-1}} \big(A_2\vec{j}.\hat{n}\big)\Big|_S. \tag{16}$$

Adding (14), (15) and (16) completes the proof of the theorem. □

Example 4.5. *Let $A_1 = x^3 - yz$, $A_2 = 2x^2y$, $A_3 = z$ and S is the surface of the cube bounded by coordinate planes $x = y = z = 0$ and the planes $x = y = z = a$.*

Then $\nabla \cdot \hat{A} = x^2 + 1$. So from (2) and (1), (13) becomes

$$\big(\ell_2\ell_3 \underset{\ell_{2,3}}{\Delta^{-1}}(x^3 - yz) - \ell_1\ell_3 \underset{\ell_{1,3}}{\Delta^{-1}} 2x^2y + \ell_1\ell_2 \underset{\ell_{1,2}}{\Delta^{-1}} z\big)\big|_S$$
$$= \ell_1\ell_2\ell_3 \underset{\ell_{1\to 3}}{\Delta^{-1}}(x^2+1)\Big|_0^a\Big|_0^a\Big|_0^a = a^3\left\{\frac{(a-\ell_1)(a-\ell_2)}{3} + \ell_1(a-\ell_1) + 1\right\}.$$

Remark 4.6. *When $\ell_1, \ell_2 \to 0, we get$*

$$\big(\ell_2\ell_3 \underset{\ell_{2\to 3}}{\Delta^{-1}}(x^3 - yz) - \ell_1\ell_3 \underset{\ell_{1,3}}{\Delta^{-1}} 2x^2y + \ell_1\ell_2 \underset{\ell_{1\to 2}}{\Delta^{-1}} z\big)\big|_S = a^3\big(\frac{a^2}{3} + 1\big).$$

5. Conclusion

In this paper, we have derived discrete convolution theorem, Green's theorem and Gauss Divergence theorem using the difference operator Δ_ℓ. The applications of the theorems are quite diverse and playing an important role in the field of electricity and magnetism.

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