J. Comp.Matha. Vol.2(1), (2018). ISSN: 2456-8686

Original Research Article

SACRED HEART RESEARCH PUBLICATIONS

Journal of Computational Mathematica

Journal homepage: www.shcpub.edu.in

ISSN: 2456-8686

Forced Oscillation of Solutions of Fractional Neutral Nonlinear Partial Differential Equations

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Received on 22 December 2017, Accepted on 2 May 2018

Abstract. In this article, we investigate the oscillation of fractional order neutral partial differential equations of the form

$$D_{+,t}^{1+\alpha}\left[u(x,t)+\lambda(t)u(x,t-\sigma)\right]+r(t)D_{+,t}^{\alpha}u(x,t)=a(t)\Delta u(x,t)$$
$$+\sum_{j=1}^{l}a_j(t)\Delta u(x,t-\tau_j)-p(x,t)F\left(\int_0^t(t-s)^{-\alpha}u(x,s)ds\right)$$
$$+f(x,t),\ (x,t)\in\Omega\times R_+=G.$$

Using the generalized Riccati technique and integral averaging method, new oscillation criteria are established.

Subject classification: 34K37, 35B05, 35R11.

1. Introduction

Neutral differential equations are functional differential equation in which the highest order derivative of the unknown function appear both with and without deviations. The neutral differential equations arise in many areas of applied mathematics. During the last two decades there has been a lot of interest towards the study of qualitative theory of neutral differential equations. A good guide concerning the literature for ordinary neutral functional differential equation [14,29] and the references cited therein. The neutral delay differential equations arise in modeling of the networks containing lossless transmission lines

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(as in high-speed switching circuits), second order neutral delay differential equations appear in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, theory of automatic control and in neuromechanical systems in which inertia plays an important role. See [16,34]. On the other hand, for partial neutral functional differential equations we refer to the reader to [15,35] and the references cited therein.The investigation of the problem of oscillation and nonoscillation of neutral delay differential equations has been initiated by [36]. Recently, researchers have established some oscillation results for partial functional equations we refer the reader to [3,11,22] for parabolic equation and to [2,4-6,19] for hyperbolic equations.

In the last few years, the problem of oscillation of solutions of fractional order differential equation have received a great deal of attention. we refer in particular to the papers [7,10,12,13]. For general back round on fractional differential equations (see the monographs [1,8,9,20,21,23,29,30,31,37]). However, it seems that very little is known regarding the oscillatory behavior of fractional order partial differential equations [18,24-27,32,33] and the references cited therein.

In [33], Wei Nian Li investigated the oscillation properties for solutions of a kind of partial fractional differential equations with damping term of the form

$$\begin{aligned} D_{+,t}^{1+\alpha}u(x,t) + p(t)D_{+,t}^{\alpha}\left(u(x,t)\right) &= a(t)\Delta u(x,t) + \sum_{i=1}^{m} a_i(t)\Delta u(x,t-\tau_i) \\ &\quad - q(x,t)\left(\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi\right),\ (x,t)\in G=\Omega\times R_+. \end{aligned}$$

It seems that there has been no attempt made on $(1+\alpha)^{th}$ order fractional neutral partial differential equation. Motivated by this, we study the following equation (1.1). In this paper, we obtain some new oscillation criteria for fractional order neutral partial differential equations with damping term of the form

$$\begin{aligned} D_{+,t}^{1+\alpha}\left[u(x,t)+\lambda(t)u(x,t-\sigma)\right] + r(t)D_{+,t}^{\alpha}u(x,t) &= a(t)\Delta u(x,t) + \sum_{j=1}^{l} a_j(t)\Delta u(x,t-\tau_j) \\ &\quad - p(x,t)F\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right) + f(x,t),\ (x,t)\in\Omega\times R_+ = G, \quad (1) \end{aligned}$$

Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega, \alpha \in (0,1)$ is a constant, $D^{\alpha}_{+,t}$ is the Riemann-Liouville fractional derivative of order α of u with respect to t and $\Delta u(x,t) = \sum_{r=1}^{n} \frac{\partial^2 u(x,t)}{\partial {x_r}^2}$ is the Laplacian operator in the Euclidean n- space R^n.

Equation (1.1) is associated with the boundary conditions, namely

$$\frac{\partial u(x,t)}{\partial \gamma} + \psi(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+, \tag{2}$$

where γ is the unit outward drawn normal vector to $\partial\Omega$ and $\psi(x,t)$ is nonnegative continuous function on $\partial\Omega \times R_+$ and

$$u(x,t) = 0, (x,t) \in \partial\Omega \times R_+. \tag{3}$$

We assume the following conditions throughout this paper without mentioning that

- (A_1) $\lambda \in C^{1+\alpha}\left([0,\infty);[0,\infty)\right), 0 \leq \lambda < 1$ and σ is a nonnegative constant;
- (A_2) $r \in C\left([0,\infty);[0,\infty)\right), a \in C\left([0,\infty);[0,\infty)\right), a_j \in C\left([0,\infty);[0,\infty)\right)$ and τ_j are nonnegative constant, $j = 1,2,3,...,l$;
- (A_3) $p \in C(\bar{G};R_+)$ and $p(t) = min_{x\in\bar{\Omega}} p(x,t)$;
- (A_4) $F \in C(R;R)$ is convex in $[0,\infty)$, and $uF(u) > 0$ for $u \neq 0$ and there exists a positive constants ρ such that $\frac{F(u)}{u} \geq \rho$ for $u \neq 0$;
- (A_5) $f \in C(\bar{G};R)$ such that $\int_{\Omega} f(x,t)dx \leq 0$.

By a solution of the problem (1.1),(1.2) (or (1.1),(1.3)), we mean a function $u(x,t) \in C^{1+\alpha}(G) \cap C^{\alpha}(\bar{G})$ which satisfies (1.1) on G and the associated boundary condition (1.2)(or (1.3)).

The solution u(x,t) of (1.1), (1.2) or (1.1), (1.3) is said to be oscillatory in the domain G if for any positive number θ there exists a point $(x_0,t_0) \in \Omega \times [\theta,\infty)$ such that $u(x_0.t_0) = 0$ holds.

Our main aim of this paper is to establish new oscillation criteria for (1.1), (1.2) and (1.1), (1.3) by using generalized Riccati technique method.

2. Preliminaries

In this section, we give the definitions of fractional derivatives and integrals which are useful throughout this paper, there are several kinds of definitions of fractional derivatives and integrals. In this paper, we use the Riemann-Liouville left sided definition on the half axis R_+.

Definition 2.1. [20] The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x,t)$ is given by

$$(D^{\alpha}_{+,t}u)(x,t) := \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\nu)^{-\alpha}u(x,\nu)d\nu \tag{4}$$

provided the right hand side is pointwise defined on R_+, where Γ is the gamma function.

Definition 2.2. [20] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : R_+ \to R$ on the half-axis R_+ is given by

$$(I^{\alpha}_{+}y)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\nu)^{\alpha-1}y(\nu)d\nu \quad \text{for} \quad t > 0 \tag{5}$$

provided the right hand side is pointwise defined on R_+.

Definition 2.3. [20] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : R_+ \to R$ on the half-axis R_+ is given by

$$(D^{\alpha}_{+}y)(t) := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\left(I^{\lceil\alpha\rceil-\alpha}_{+}y\right)(t) \quad \text{for} \quad t > 0 \tag{6}$$

provided the right hand side is pointwise defined on R_+, where $\lceil\alpha\rceil$ is the ceiling function of α.

We need the following lemmas in proving our main results.

Lemma 2.4. *[24] Let*

$$K(t) := \int_0^t (t-\nu)^{-\alpha}y(\nu)d\nu \quad \textit{for} \quad \alpha\in(0,1) \quad \textit{and} \quad t > 0. \tag{7}$$

Then $K'(t) = \Gamma(1-\alpha)(D^{\alpha}_{+}y)(t)$.

Lemma 2.5. *[20] Let $\alpha \geq 0, m \in N$ and $D = \frac{d}{dt}$. If the fractional derivatives $(D_+^{\alpha} y)(t)$ and $(D_+^{\alpha+m} y)(t)$ exists*

$$D^m((D_+^{\alpha} y)(t)) = (D_+^{\alpha+m} y)(t). \tag{8}$$

Lemma 2.6. *[17] If X and Y are nonnegative, then*

$$mXY^{m-1} - X^m \leq (m-1)Y^m. \tag{9}$$

3. Main Results

In this section, we establish the oscillation of the problem (1.1), (1.2). We start with the following theorem.

Theorem 3.1. *Suppose that the conditions $(A_1) - (A_5)$ hold. Assume that for some $t_0 > 0$ and $F'(v)$ exists such that $F'(v) \geq \mu$ for some $\mu > 0$ and for all $v \neq 0$,*

$$\int_{t_0}^{\infty} \exp\left(\int_{t_0}^{t} r(s)ds\right) dt = \infty, \tag{10}$$

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[v(s)p(s) - \frac{(r(s))^2 v(s)}{4\mu\Gamma(1-\alpha)}\right] ds = \infty, \tag{11}$$

where

$$v(t) = \exp\left(-\int_{t_0}^{t} r(s)ds\right), t \geq t_0. \tag{12}$$

Then every solution of the problem (1.1), (1.2) is oscillatory in G.

Proof. Let us prove this theorem by the method of contradiction. Assume that there is a non-oscillatory solution $u(x,t)$ to the problem (1.1), (1.2) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality we may suppose that $u(x,t) > 0, u(x, t-\tau_j) > 0$ and $u(x, t-\sigma) > 0$ in $\Omega \times [t_1, \infty)$ for $t_1 \geq t_0$, $j = 1, 2, 3, ..., l$. Integrating (1.1) over the domain Ω, we get

$$\int_\Omega D_{+,t}^{1+\alpha}\left[u(x,t)+\lambda(t)u(x,t-\sigma)\right]dx + r(t)\int_\Omega D_{+,t}^{\alpha}u(x,t)dx = a(t)\int_\Omega \Delta u(x,t)dx$$
$$+\sum_{j=1}^{l}a_j(t)\int_\Omega \Delta u(x,t-\tau_j)dx - \int_\Omega p(x,t)F\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx$$
$$+\int_\Omega f(x,t)dx,\ t\geq t_1. \tag{13}$$

Using Green's formula and (1.2), it follows

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega}\frac{\partial u(x,t)}{\partial\gamma}dS = -\int_{\partial\Omega}\psi(x,t)u(x,t)dS \leq 0, \quad t\geq t_1 \tag{14}$$

and

$$\int_\Omega \Delta u(x,t-\tau_j)dx = \int_{\partial\Omega}\frac{\partial u(x,t-\tau_j)}{\partial\gamma}dS$$

$$= -\int_{\partial\Omega}\psi(x,t-\tau_j)u(x,t-\tau_j)dS \leq 0, \quad t\geq t_1,\ j=1,2,3,...,l. \tag{15}$$

where dS is the surface element on $\partial\Omega$.

$$Let\ V(t) = \int_\Omega u(x,t)dx. \tag{16}$$

By using Jensen's inequality and (A_3), we obtain

$$\begin{aligned}\int_\Omega p(x,t)F\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx &\geq p(t)F\left(\int_\Omega\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx\right)\\ &\geq p(t)F\left(\int_0^t (t-s)^{-\alpha}\left(\int_\Omega u(x,s)dx\right)ds\right)\\ &\geq p(t)F\left(K(t)\right), \quad t\geq t_1. \end{aligned}\tag{17}$$

By (A_5),

$$\int_\Omega f(x,t)dx \leq 0. \tag{18}$$

Combining $(3.5)-(3.9)$, we have

$$D_+^{1+\alpha}\left[V(t)+\lambda(t)V(t-\sigma)\right] + r(t)D_+^{\alpha}V(t) \leq -p(t)F\left(K(t)\right), \quad t\geq t_1.$$

Let

$$z(t) = V(t) + \lambda(t)V(t-\sigma). \tag{19}$$

Then the above inequality becomes,

$$D_+^{1+\alpha}[z(t)] + r(t)D_+^{\alpha}V(t) + p(t)F(K(t)) \le 0, \quad t \ge t_1. \tag{20}$$

Using Lemma 2.2 and (3.11), we have

$$\begin{aligned}\left[\left(D_+^{\alpha}z(t)\right)v(t)\right]' &= D_+^{1+\alpha}(z(t))\,v(t) - \left(D_+^{\alpha}z(t)\right)r(t)v(t)\\ &= \left[-r(t)\left(D_+^{\alpha}V(t)\right) - p(t)F(K(t))\right]v(t) - \left(D_+^{\alpha}z(t)\right)r(t)v(t)\end{aligned} \tag{21}$$

$$< 0, \quad t \ge t_1.$$

Therefore, $\left(D_+^{\alpha}z(t)\right)v(t)$ is a strictly decreasing in $[t_1,\infty)$. We claim that $D_+^{\alpha}z(t) > 0$ for $t \ge t_1$. If not, then $D_+^{\alpha}z(t) \le 0$ for $t \ge t_1$. Therefore, there exists a $T \ge t_1$ such that

$$\left(D_+^{\alpha}z(t)\right)v(t) < \left(D_+^{\alpha}z(T)\right)v(T) := -C < 0, \quad t > T. \tag{22}$$

Using Lemma 2.1 in (3.13), we get

$$\frac{K'(t)}{\Gamma(1-\alpha)} = D_+^{\alpha}z(t) < \frac{-C}{v(t)} = -C\exp\left(\int_{t_0}^{t} r(s)ds\right)$$

$$\exp\left(\int_{t_0}^{t} r(s)ds\right) < -\frac{K'(t)}{C\Gamma(1-\alpha)}, \quad t \ge T. \tag{23}$$

Now, we integrating (3.14) from T to t, we have

$$\begin{aligned}\int_{T}^{t}\exp\left(\int_{t_0}^{t} r(s)ds\right)dt &< \frac{-K(t)+K(T)}{C\Gamma(1-\alpha)}\\ &< \frac{K(T)}{C\Gamma(1-\alpha)} := K_1.\end{aligned} \tag{24}$$

Letting $t \to \infty$, which contradicts to (3.1). Hence $D_+^\alpha z(t) \geq 0$ for $t \geq t_1$ holds. We define the generalized Riccati function $w(t)$ by

$$w(t) = \frac{\left(D_+^\alpha z(t)v(t)\right)}{F\left(K(t)\right)}, \quad t \geq t_1. \tag{25}$$

we have $w(t) > 0$ and

$$\begin{aligned} w'(t) &= \frac{\left((D_+^\alpha z(t))v(t)\right)'}{F\left(K(t)\right)} - \frac{\left(D_+^\alpha z(t)\right)v(t)F'\left(K(t)\right)K'(t)}{F^2\left(K(t)\right)} \\ &\leq \frac{(D_+^{1+\alpha} z(t))v(t) - (D_+^\alpha z(t))v(t)r(t)}{F\left(K(t)\right)} - \frac{\mu\Gamma(1-\alpha)D_+^\alpha z(t)v(t)D_+^\alpha z(t)}{F^2\left(K(t)\right)} \\ &\leq \frac{-r(t)(D_+^\alpha V(t))w(t)}{D_+^\alpha z(t)} - p(t)v(t) - w(t)r(t) - \frac{\mu\Gamma(1-\alpha)w^2(t)}{v(t)} \\ &\leq -p(t)v(t) - w(t)r(t) - \frac{\mu\Gamma(1-\alpha)w^2(t)}{v(t)} \end{aligned} \tag{26}$$

Taking m=2,

$$X = \sqrt{\frac{\mu\Gamma(1-\alpha)}{v(t)}}w(t), \quad Y = \frac{r(t)\sqrt{v(t)}}{2\sqrt{\mu\Gamma(1-\alpha)}}. \tag{27}$$

Using Lemma 2.3 and (3.18) in (3.17), we get

$$w'(t) \leq -p(t)v(t) + \frac{\left(r(t)\right)^2 v(t)}{4\mu\Gamma(1-\alpha)}.$$

Integrating both sides of the above inequality from t_1 to t, we have

$$\begin{aligned} \int_{t_1}^{t} \left[p(s)v(s) - \frac{\left(r(s)\right)^2 v(s)}{4\mu\Gamma(1-\alpha)}\right] ds &\leq -\int_{t_1}^{t} w'(s)ds \\ &= w(t_1) - w(t) \\ &< w(t_1) \end{aligned}$$

Taking the limit supremum of both sides of the above inequality as t$\to \infty$ we get

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[p(s)v(s) - \frac{\left(r(s)\right)^2 v(s)}{4\mu\Gamma(1-\alpha)}\right] ds < w(t_1) < \infty$$

which contradicts (3.2). This completes the proof of Theorem 3.1. $\square$

Theorem 3.2. *Assume that* $(A_1)-(A_5)$, (3.1) *hold and*

$$\int_{t_0}^{\infty}\left[\rho\ p(s)v(s)-\frac{v(s)\left(r(s)\right)^2}{4\Gamma(1-\alpha)}\right]ds=\infty,\quad t_0>0. \tag{28}$$

Then every solution of the problem (1.1), (1.2) *is oscillatory in* G.

Proof. Assume that there is a non-oscillatory solution $u(x,t)$ to the problem (1.1), (1.2) which has no zero in $\Omega\times[t_0,\infty)$ for some $t_0>0$. Without loss of generality we may suppose that $u(x,t)>0$, $u(x,t-\tau_j)>0$ and $u(x,t-\sigma)>0$ in $\Omega\times[t_1,\infty)$ for $t_1\geq t_0$, $j=1,2,3,...,l$. Proceeding as in the proof of Theorem 3.1, we get (3.11) and $D_+^{\alpha}z(t)>0$ for $t\geq t_1$.
Define the function as follows, $w_1(t)$ by

$$w_1(t)=\frac{D_+^{\alpha}z(t)v(t)}{K(t)},\quad t\geq t_1. \tag{29}$$

Then, $w_1(t)>0$,

$$\begin{aligned}
w_1'(t)&=\frac{\left((D_+^{\alpha}z(t))v(t)\right)'}{K(t)}-\frac{\left(D_+^{\alpha}z(t)\right)v(t)K'(t)}{K^2(t)}\\
&=\frac{(D_+^{1+\alpha}z(t))v(t)-(D_+^{\alpha}z(t))v(t)r(t)}{K(t)}-\frac{\Gamma(1-\alpha)D_+^{\alpha}z(t)v(t)D_+^{\alpha}z(t)}{K^2(t)}\\
&\leq\frac{-r(t)(D_+^{\alpha}V(t))v(t)}{K(t)}-\rho\ p(t)v(t)-w_1(t)r(t)-\frac{\Gamma(1-\alpha){w_1}^2(t)}{v(t)} \qquad (30)\\
&\leq-\rho\ p(t)v(t)-w_1(t)r(t)-\frac{\Gamma(1-\alpha){w_1}^2(t)}{v(t)}\\
&\leq-\rho\ p(t)v(t)+\frac{v(t)(r(t))^2}{4\Gamma(1-\alpha)},\quad t\geq t_1. \qquad (31)
\end{aligned}$$

By integrating (3.22) from t_1 to t, we have

$$\begin{aligned}
\int_{t_1}^{t}\left[\rho\ p(s)v(s)-\frac{v(s)(r(s))^2}{4\Gamma(1-\alpha)}\right]ds&\leq-\int_{t_1}^{t}w_1'(s)ds\\
&=w_1(t_1)-w_1(t)\\
&=w_1(t_1)<\infty,
\end{aligned}$$

which contradicts (3.19). Hence the proof is complete. □

Next, we will discuss some new oscillation criteria for (1.1) by using integral averaging conditions of Philo's type.

Theorem 3.3. *Suppose that the conditions $(A_1)-(A_5)$ hold. Furthermore, suppose that there exists a function $H \in$ C(D,R) where D:=$\{(t,s) : t \geq s \geq t_0\}$ such that*
1.$H(t,t) = 0$ *for* $t \geq t_0$,
2.$H(t,s) > 0$ *for* $(t,s) \in D_0$, *where* $D_0 = \{(t,s) : t > s \geq t_0\}$.
H has a continuous and non positive partial derivative $H'_s(t,s) = \frac{\partial H(t,s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s) \left[\rho\, p(s)v(s) - \frac{v(s)\,(r(s))^2}{4\Gamma(1-\alpha)}\right] ds = \infty. \tag{32}$$

Then all solutions of (1.1), (1.2) is osillatory in G.

Proof. Suppose that u(x,t) is a non-oscillatory solution of (1.1), (1.2). Without loss of generality, we may suppose that u(x,t) is an eventually positive solution of (1.1), (1.2). We proceeding as in the proof of Theorem 3.2, to get that (3.22)

$$w_1'(t) \leq \left[-\rho\, p(t)v(t) + \frac{v(t)\,(r(t))^2}{4\Gamma(1-\alpha)}\right]$$

Multiplying the previous inequality by H(t,s) and integrating from t_1 to t for t $\in [t_0, \infty)$, we obtain

$$\int_{t_1}^{t} H(t,s) \left[\rho\, p(s)v(s) - \frac{v(s)\,(r(s))^2}{4\Gamma(1-\alpha)}\right] ds \leq -\left[H(t,s)w_1(s)\right]_{t_1}^{t} + \int_{t_1}^{t} H'_s(t,s)w_1(s)ds$$
$$< H(t,t_1)w_1(t_1).$$

Therefore,

$$\frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s) \left[\rho\, p(s)v(s) - \frac{v(s)(r(s))^2}{4\Gamma(1-\alpha)}\right] ds < w_1(t_1)$$
$$< \infty$$

which is contradiction to (3.23). The proof is complete. □

4. Oscillation of the problem (1.1), (1.3)

In this section we establish some new criteria for the oscillations of the solutions of the equation (1.1),(1.3). For this we need the following.
The smallest eigenvalue β_0 of the Dirichlet problem

$$\Delta\omega(x) + \beta\omega(x) = 0 \quad \text{in} \quad \Omega$$
$$\omega(x) = 0 \quad \text{on} \quad \partial\Omega,$$

is positive and the corresponding eigen function $\phi(x)$ is positive in Ω.

Theorem 4.1. *Suppose that the conditions of Theorem 3.1 hold. Then every solution of the problem (1.1),(1.3) is oscillatory in G.*

Proof. Assume to the contrary that there is a non-oscillatory solution $u(x,t)$ to the problem $(1.1), (1.3)$ which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality we may suppose that $u(x,t) > 0,\ u(x, t-\tau_j) > 0$ and $u(x, t-\sigma) > 0$ in $\Omega \times [t_1, \infty)$ for $t_1 \geq t_0,\ j = 1, 2, 3, ..., l$. Multiplying both sides of (1.1) by $\phi(x)$ and integrating over the domain Ω, we get

$$\begin{aligned}
&\int_\Omega D^{1+\alpha}_{+,t}\left[u(x,t) + \lambda(t)u(x,t-\sigma)\right]\phi(x)dx + r(t)\int\limits_\Omega D^{\alpha}_{+,t}u(x,t)\phi(x)dx \\
&= a(t)\int_\Omega \Delta u(x,t)\phi(x)dx + \sum_{j=1}^{l} a_j(t)\int\limits_\Omega \Delta u(x,t-\tau_j)\phi(x)dx \\
&- \int\limits_\Omega p(x,t)F\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)\phi(x)dx + \int\limits_\Omega f(x,t)\phi(x)dx, \quad t \geq t_1. \quad (33)
\end{aligned}$$

Using Green's formula and (1.3) we get,

$$\begin{aligned}
\int_\Omega \Delta u(x,t)\phi(x)dx &= \int_\Omega u(x,t)\Delta\phi(x)dx \\
&= -\beta_0\int_\Omega u(x,t)\phi(x)dx \leq 0, \quad t \geq t_1 \qquad (34)
\end{aligned}$$

and

$$\int_\Omega \Delta u(x,t-\tau_j)\phi(x)dx = \int_\Omega u(x,t-\tau_j)\Delta\phi(x)dx$$

$$= -\beta_0 \int_\Omega u(x,t-\tau_j)\phi(x)dx \leq 0, \quad t \geq t_1,\ j = 1,2,3,...,l. \quad (35)$$

Let

$$U(t) = \int_\Omega u(x,t)\phi(x)dx, \quad t \geq t_1. \quad (36)$$

By using Jensen's inequality and (A_3), we get

$$\begin{aligned}
&\int_\Omega p(x,t)F\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)\phi(x)dx \\
&\geq p(t)F\left(\int_\Omega \left(\int_0^t (t-s)^{-\alpha}u(x,s)\phi(x)ds\right)dx\right) \\
&\geq p(t)F\left(\int_0^t (t-s)^{-\alpha}\left(\int_\Omega u(x,s)\phi(x)dx\right)ds\right) \\
&\geq p(t)F\left(K(t)\right), \quad t \geq t_1. \qquad (37)
\end{aligned}$$

In view of $(4.2)-(4.5), (4.1)$ yields,

$$D_+^{1+\alpha}\left[z(t)\right] + r(t)D_+^\alpha U(t) + p(t)F\left(K(t)\right) \leq 0,\ t \geq t_1. \quad (38)$$

The rest of the proof is similar to that of the Theorem 3.1. □

The following Theorems can be proved analogously.

Theorem 4.2. *Suppose that the conditions of Theorem 3.2 hold. Then every solution of the problem (1.1),(1.3) is oscillatory in G.*

Theorem 4.3. *Suppose that the conditions of Theorem 3.3 hold. Then every solution of the problem (1.1),(1.3) is oscillatory in G.*

5. Example

In this section, we give some examples to illustrate our main theorem in section 3.

Example 5.1. *We consider the fractional partial differential equation*

$$D_{+,t}^{1+\frac{1}{2}}\left[u(x,t)+\frac{1}{9}u(x,t-\frac{\pi}{2})\right]+\frac{1}{t}D_{+,t}^{\frac{1}{2}}u(x,t)=\frac{8}{9\sqrt{2}}\Delta u(x,t)+\frac{1}{t}\Delta u(x,t-\frac{3\pi}{4})$$
$$-\frac{1}{\sqrt{2\pi}\left[\cos tC(x)+\sin tS(x)\right]}\int_0^t(t-s)^{\frac{-1}{2}}u(x,s)ds-\frac{10}{9\sqrt{2}}\sin x\sin t+1$$
$$for\ (x,t)\in G,\ where\ G=(0,\pi)\times[0,\infty), \tag{39}$$

with the boundary condition

$$u(0,t)=u(\pi,t)=0,\quad t\geq 0. \tag{40}$$

Take $\alpha=\frac{1}{2}, n=1, m=1, \lambda(t)=\frac{1}{9}, \sigma=\frac{\pi}{2}, r(t)=\frac{1}{t}, \tau_1=\frac{3\pi}{4}, a(t)=\frac{8}{9\sqrt{2}}, a_1(t)=\frac{1}{t}, \mu=1, p(x,t)=\frac{1}{\sqrt{2\pi}[\cos tC(x)+\sin tS(x)]}$,
where C(x) and S(x) are the Fresnel integrals namely,

$$C(x)=\int_0^x\cos\left(\frac{1}{2}\pi t^2\right)dt, S(x)=\int_0^x\sin\left(\frac{1}{2}\pi t^2\right)dt,$$

$F(u)=u$ *and* $f(x,t)=-\frac{10}{9\sqrt{2}}\sin x\sin t+1, t>\sin^{-1}(\frac{9\sqrt{2}\pi}{20}),\ \mu=1.$
But $|C(x)|\leq\pi$ *and* $|S(x)|\leq\pi$

$$p(t)=min_{x\in\bar{\Omega}}=\frac{1}{\pi\sqrt{2\pi}\left[\cos t+\sin t\right]}.$$

It is clear that conditions $(A_1)-(A_5)$ *hold.* $\int_{t_0}^{\infty}\exp(\int_{t_0}^{t}\frac{1}{s}ds)dt=\infty$ *and* $v(t)=\frac{t_0}{t}$.
Consider,

$$\int_{t_1}^{t}\left[v(s)p(s)-\frac{(r(s))^2v(s)}{4\mu\Gamma(1-\alpha)}\right]ds=\int_{t_1}^{t}\left[\frac{t_0}{s}\frac{1}{\pi\sqrt{2\pi}}\frac{1}{[\cos s+\sin s]}-\frac{1}{s^3}\frac{t_0}{4\Gamma(\frac{1}{2})}\right]ds$$
$$>\int_{t_1}^{t}\left[\frac{t_0}{s}\frac{1}{\pi\sqrt{2\pi}}-\frac{1}{s^3}\frac{t_0}{4\Gamma(\frac{1}{2})}\right]ds\to\infty\ as\ t\to\infty$$

It is easy to see that all conditions of Theorem 3.1 are satisfied. Hence every solutions of equations $(5.1),(5.2)$ *oscillates in* $(0,\pi)\times[0,\infty)$. *In fact* $u(x,t)=\sin x\cos t$ *is such a solution.*

Example 5.2.

$$D_{+,t}^{1+\frac{1}{2}}\left[u(x,t)+\frac{1}{t^{\frac{1}{4}}}u(x,t-2\pi)\right]+\frac{1}{t}D_{+,t}^{\frac{1}{2}}u(x,t)$$
$$=\left[\frac{3}{4}\frac{\sqrt{2}\pi}{(\Gamma(\frac{1}{4}))^2}t^{\frac{-7}{4}}+\frac{1}{4\sqrt{2}}t^{\frac{-5}{4}}+\frac{t^{\frac{-1}{4}}}{\sqrt{2}}\right]\Delta u(x,t)$$
$$+\left[\frac{1}{\sqrt{2}}+\frac{\sqrt{2}\pi}{(\Gamma(\frac{1}{4}))^2}t^{\frac{-3}{4}}+\frac{t^{\frac{-1}{4}}}{\sqrt{2}}\right]\Delta u(x,t-\frac{\pi}{2})$$
$$-\frac{\sin t}{2\sqrt{\pi}\left[\sin tC(x)-\cos tS(x)\right]}\int_0^t(t-s)^{\frac{-1}{2}}u(x,s)ds-\frac{1}{4\sqrt{2}}t^{\frac{-5}{4}}\cos x\sin t$$
$$+\frac{\cos x}{t\sqrt{2}}[\sin t+\cos t]\ for\ (x,t)\in G,\ where\ G=(0,\pi)\times[0,\infty), \tag{41}$$

with the boundary condition

$$u_x(0,t)=u_x(\pi,t)=0,\quad t\geq 0. \tag{42}$$

Let $\alpha=\frac{1}{2},n=1,m=1,\lambda(t)=\frac{1}{t^{\frac{1}{4}}},\sigma=2\pi,r(t)=\frac{1}{t},\tau_1=\frac{\pi}{2},$
$a(t)=\left[\frac{3}{4}\frac{\sqrt{2}\pi}{(\Gamma(\frac{1}{4}))^2}t^{\frac{-7}{4}}+\frac{1}{4\sqrt{2}}t^{\frac{-5}{4}}+\frac{t^{\frac{-1}{4}}}{\sqrt{2}}\right],a_1(t)=\left[\frac{1}{\sqrt{2}}+\frac{\sqrt{2}\pi}{(\Gamma(\frac{1}{4}))^2}t^{\frac{-3}{4}}+\frac{t^{\frac{-1}{4}}}{\sqrt{2}}\right],$
$p(x,t)=\frac{\sin t}{2\sqrt{\pi}[\sin tC(x)-\cos tS(x)]}$
where C(x) and S(x) are the Fresnel integrals namely,

$$C(x)=\int_0^x\cos\left(\frac{1}{2}\pi t^2\right)dt, S(x)=\int_0^x\sin\left(\frac{1}{2}\pi t^2\right)dt,$$

$F(u)=u$ *and*
$f(x,t)=-\frac{1}{4\sqrt{2}}t^{\frac{-5}{4}}\cos x\sin t+\frac{\cos x}{t\sqrt{2}}[\sin t+\cos t],\ \mu=1.$
It is easy to see that

$$p_1(t)=min_{x\in\bar{\Omega}}=min_{x\in[0,\pi]}=\frac{\sin t}{2\sqrt{\pi}\left[\sin tC(x)-\cos tS(x)\right]},$$

$v(t)=\frac{t_0}{t}$. *But* $|C(x)|\leq\pi$ *and* $|S(x)|\leq\pi$. *Therefore,* $p_1(t)=\frac{\sin t}{2\pi\sqrt{\pi}[\sin t-\cos t]}$.
Consider,

$$\int_{t_1}^t\left[v(s)p(s)-\frac{(r(s))^2v(s)}{4\mu\Gamma(1-\alpha)}\right]ds=\int_{t_1}^t\left[\frac{t_0}{s}\frac{\sin s}{2\pi\sqrt{\pi}}\frac{1}{[\sin s-\cos s]}-\frac{1}{s^3}\frac{t_0}{4\Gamma(\frac{1}{2})}\right]ds$$
$$>\int_{t_1}^t\left[\frac{t_0}{s}\frac{1}{4\pi\sqrt{\pi}}-\frac{1}{s^3}\frac{t_0}{4\Gamma(\frac{1}{2})}\right]ds\to\infty\ as\ t\to\infty$$

Thus all the conditions of Theorem (3.1) are satisfied. Then every solution of the problem (5.3),(5.4) oscillates in $(0,\pi)\times[0,\infty)$. *Infact* $u(x,t)=\cos x\sin t$ *is one such solution of (5.3), (5.4).*

Example 5.3. *we consider the fractional partial differential equation*

$$\begin{aligned}
&D^{1+\frac{1}{2}}_{+,t}\left[u(x,t)+\frac{1}{2}u(x,t-\frac{\pi}{2})\right]+D^{\frac{1}{2}}_{+,t}u(x,t)=2\Delta u(x,t)+3\Delta u(x,t-\pi)\\
&-\frac{1}{4t}\left(\int_0^t(t-s)^{\frac{-1}{2}}u(x,s)ds\right)^2-\frac{3x}{4\sqrt{\pi}}\frac{1}{t^{\frac{3}{2}}}+\frac{x}{\sqrt{\pi t}}+x^2\\
&for\ (x,t)\in G,\ where\ G=(0,\pi)\times[0,\infty),
\end{aligned}\tag{43}$$

with the boundary condition

$$u(0,t)=u(\pi,t)=0,\qquad t\geq 0.\tag{44}$$

Here $\alpha=\frac{1}{2}$, $n=1$, $m=1$, $\lambda(t)=\frac{1}{2}$, $\sigma=\frac{\pi}{2}$, $r(t)=1$, $\tau_1=\pi$, $a(t)=2$, $a_1(t)=3$, $p(x,t)=\frac{1}{4t}$ $F(u)=u^2$ *and* $f(x,t)=-\frac{3x}{4\sqrt{\pi}}\frac{1}{t^{\frac{3}{2}}}+\frac{x}{\sqrt{\pi t}}+x^2$, *where* $t>\frac{3^{\frac{4}{3}}}{4\pi}$, $\mu=1$. *It is clear that conditions* $(A_1)-(A_5)$ *hold.* $\int_{t_0}^{\infty}\exp(\int_{t_0}^{t}ds)dt=\infty$ *and* $v(t)=e^{t_0-t}$. *Consider,*

$$\int_{t_1}^{t}\left[v(s)p(s)-\frac{(r(s))^2v(s)}{4\mu\Gamma(1-\alpha)}\right]ds=\int_{t_1}^{t}\left[\frac{e^{t_0}e^{-s}}{2\sqrt{t}}-\frac{e^{t_0}e^{-s}}{4\sqrt{\pi}}\right]<\infty$$

It is easy to see that conditions of Theorem 3.1 are not satisfied. Hence every solutions of equations (5.5), (5.6) *need not oscillate in* $(0,\pi)\times[0,\infty)$. *In fact,* $u(x,t)=x$ *is not a oscillatory solution.*

Remark 1 *The results obtain in this paper can be considered as generalization of those results in paper [33]. In fact,* $\lambda(t)=0, F(u)=u, f(x,t)=0$. *Our equation (1.1) reduces to equation (2.1) in [33].*

Remark 2 *Note that the results by [33] cannot be applied to the equations (5.1),(5.3) and (5.5). So our results improve the results in [33].*

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