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# Higher Order Fibonacci Summation Formula with Extorial Functions

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Abstract. This paper aims to obtain extorial type solutions of Fibonacci difference equations having shift value. A higher order Fibonacci summation formula of product of polynomial and extorial functions is obtained by higher order Fibonacci nabla difference operator, its inverse and the higher order Fibonacci numbers. Extorial function is a function obtained by replacing polynomial into polynomial factorials in the exponential function. Suitable examples with numerical verification are inserted to illustrate our main results.

**Key words:** Exact solution, Extorial function,Fibonacci difference equation, Fibonacci numbers, Nabla difference operator, Summation formula.

**Mathematics Subject classification 2010:** $47B39$, $39A70$, $11J54$, $33B15$.

## 1. Introduction

Difference equations usually describe the evolution of certain phenomena over the course of time. A standard approach in numerical integration of differential equations is to replace it by a suitable difference equation whose solutions can be obtained in a stable manner. However, the qualitative properties of solutions of the difference equations are quite different from the solutions of the corresponding differential equations. Further, solutions of several well known difference equations like Clairaut's, Euler's, Riccati's, Bernoulli's, Verhulst's, Duffing's, Mathieu's and Volterra's difference equations retain most of the properties of the corresponding differential equations.

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A difference equation is an equation that contains sequence of differences. We can solve a difference equation by finding a sequence that satisfies the equation and we call that sequence a solution of the equation. A function satisfying given difference equation is called exact solution of the given difference equation.For example, the difference equation $\Delta u(k) = e^{sk}$ has an exact solution $u(k) = e^{sk}/(e^s - 1),\ s \neq 0$ and $k \in (-\infty, \infty)$. As extensions of $\Delta$, theories of q-difference, h-difference and fractional difference operators are found in [1]- [3]

In mathematical terms, the sequence $F_n$ of usual Fibonacci numbers is obtained by the recurrence relation $F_n = F_{n-1} + F_{n-2},\ F_0 = 1,\ F_1 = 1,\ n \geq 2$
In [7], for an m-tuple $\overline{a} = (a_1, a_2, a_3, \cdots a_m) \in \mathbb{R}^m$, $\overline{a}$-Fibonacci number is defined as

$$F_n(\overline{a}) = \sum_{i=1}^{n} a_i F(n-i), F(0) = 1, F(-n) = 0, \qquad 1 \leq n < m \tag{1}$$
$$F_n(\overline{a}) = a_1 F_{n-1} + a_2 F_{n-2} + \cdots + a_m F_{n-m}, \qquad n \geq m$$

*Remark* 1.1. For our convenient, we denote $F_n(\overline{a}) = \overline{F}_n$.

## 2. PRELIMINARIES

In this section, the exponential function is extended to extorial function and applying by higher order $\overline{a}$-Fibonacci number, we obtain higher order summation formula for product of extorial function and polynomials.

**Definition 2.1.** [4] Let $u(k), k \in (-\infty, \infty)$, be a real or complex valued function and $\ell > 0$ be fixed. Then, the $\ell$-difference operator $\Delta_\ell$ on $u(k)$ is defined as

$$\Delta_\ell u(k) = u(k+\ell) - u(k). \tag{2}$$

and its inverse is defined as if there is a function $v(k)$ such that

$$\Delta_\ell v(k) = u(k), \quad then \quad v(k) = \Delta_\ell^{-1} u(k). \tag{3}$$

**Definition 2.2.** [6] For $0 \neq \ell, k \in (-\infty, \infty)$ and $n \in \mathbb{N}(0)$, the $\ell-$polynomial factorial is defined as

$$k_\ell^{(n)} = k(k-\ell)(k-2\ell)\ldots(k-(n-1)\ell). \tag{4}$$

For any real number, the $\nu^{th}$ order polynomial factorial is given by

$$k_\ell^\nu = \ell^\nu \left[ \frac{\Gamma\left(\frac{k}{\ell}+1\right)}{\Gamma\left(\frac{k}{\ell}+1-\nu\right)} \right], \quad \frac{k}{\ell}+1-\nu \notin -N(0)$$

Where $\Gamma$ is the Gamma function.

**Definition 2.3.** [6] For $-1 < \ell < 1$ and $k \in (-\infty, \infty)$, the $\ell-$extorial function, denoted as $e(k_\ell)$, is defined as

$$e(k_\ell) = \frac{k_\ell^{(0)}}{0!} + \frac{k_\ell^{(1)}}{1!} + \frac{k_\ell^{(2)}}{2!} + \frac{k_\ell^{(3)}}{3!} + \ldots + \infty. \tag{5}$$

In general, for any real $\nu$, we have

$$e_\nu(k_\ell) = \frac{k_\ell^{(0)}}{0!} + \frac{k_\ell^{(\nu)}}{(\nu)!} + \frac{k_\ell^{(2\nu)}}{(2\nu)!} + \frac{k_\ell^{(3\nu)}}{(3\nu)!} + \ldots + \infty.$$

*Remark* 2.4. The additive property of extorial function is given by

$$e_1(k_1+k_2)_\ell = e_1((k_1)_\ell)e_1((k_2)_\ell) \tag{6}$$

**Definition 2.5.** For $a = (a_1, a_2, a_3, \cdots, a_m) \in \mathbb{R}^m$, the Fibonacci Nabla $\ell-$difference operator $\underset{(a)\ell}{\nabla}$ on u(k) is defined as

$$\underset{(a)\ell}{\nabla} v(k) = v(k) - a_1 v(k+\ell) - a_2 v(k+2\ell) - a_3 v(k+3\ell) - \cdots - a_m v(k+m\ell). \tag{7}$$

and its inverse is given by, if

$$\underset{(a)\ell}{\nabla} v(k) = u(k) \qquad then \qquad v(k) = \underset{(a)\ell}{\overset{-1}{\nabla}} u(k), \tag{8}$$

**Lemma 2.6.** *(Product Formula) Let* $1 - \sum_{j=1}^{m} a_j e(j\ell_\ell) \neq 0$, *From* (7) *and*(6), *we have*

$$\underset{(a)\ell}{\nabla} e(k_\ell) = e(k_\ell) - a_1 e((k+\ell)_\ell) - a_2 e((k+2\ell)_\ell) - \cdots - a_m e((k+m\ell)_\ell). \tag{9}$$

$$= e(k_\ell) - a_1 e(k_\ell)e(\ell_\ell) - a_2 e(k_\ell)e(2\ell_\ell) - \cdots - a_m e(k_\ell)e(m\ell_\ell)$$

$$= e(k_\ell)[1 - a_1 e(\ell_\ell) - a_2 e(2\ell_\ell) - \cdots - a_m e(m\ell_\ell)],$$

*which yields*

$$\underset{(a)\ell}{\overset{-1}{\nabla}} e(k_\ell) = \frac{e(k_\ell)}{1 - \sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)}. \tag{10}$$

**Theorem 2.7.** *(Fibonacci Summation Formula)Consider $\overline{F}_n$ given in* (1) *and* (2). *Let $v(k)$ be a solution of the higher order difference equation $\underset{(a)\ell}{\nabla} v(k) = u(k)$, $k \in [0,\infty)$, then we have*

$$v(k) - \sum_{j=1}^{m}\sum_{i=j}^{m} a_i F_{n-(i-j)} v(k+(n+j)\ell) = \sum_{i=0}^{n} F_i u(k+i\ell). \tag{11}$$

*Proof.* From (7) and (8), we get

$$v(k) = u(k) + a_1 v(k+\ell) + a_2 v(k+2\ell) + \cdots + a_m v(k+m\ell). \tag{12}$$

Replacing $k$ by $k+\ell$ and then substituting the value $v(k+\ell)$ in (12), we find
$v(k) = u(k) + a_1 u(k+\ell) + (a_1^2 + a_2) v(k+2\ell) + (a_1 a_2 + a_3) v(k+3\ell)$

$$+ \cdots + (a_1 a_{m-1} + a_m) v(k+m\ell) + a_1 a_m v(k+(m+1)\ell). \tag{13}$$

Replacing $k$ by $k+2\ell$ and then substituting the value $v(k+2\ell)$ in (13), we obtain

$$v(k) = u(k) + a_1 u(k+\ell) + (a_1^2 + a_2) u(k+2\ell) + ((a_1^2 + a_2)a_1 + a_1 a_2 + a_3) v(k+3\ell)$$

$$+ \cdots + ((a_1^2 + a_2) a_{(m-2)} + a_1 a_{(m-1)} + a_m) v(k+m\ell)$$

$$+ ((a_1^2 + a_2) a_{m-1} + a_1 a_m) v(k+(m+1)\ell) + (a_1^2 + a_2) a_m v(k+(m+2)\ell),$$

which can be expressed as

$$\begin{aligned} v(k) = {} & F_0 u(k) + F_1 u(k+\ell) + F_2 u(k+2\ell) + F_3 v(k+3\ell) \\ & + \sum_{i=2}^{m-2} (F_2 a_i + F_1 a_{i+1} + F_0 a_{i+2}) v(k+(i+2)\ell) \\ & + (F_2 a_{m-1} + F_1 a_m) v(k+(m+1)\ell) + F_2 a_m v(k+(m+2)\ell) \end{aligned}$$

Repeating this process again and again and by induction, we arrive (11). $\square$

**Corollary 2.8.** *If* $\sum_{j=1}^{m} a_j e(j\ell_\ell) \neq 1$, *then we have*

$$\frac{e(k_\ell) - \sum_{j=1}^{m}\sum_{i=j}^{m} a_i F_{n-(i-j)} e(k+(n+j)\ell)}{1-\sum_{j=1}^{m} a_j e(j\ell_\ell)} = \sum_{i=0}^{n} F_i e((k+i\ell)_\ell). \tag{14}$$

*Proof.* Taking $u(k) = e(k_\ell)$ and applying (10) yield

$$\underset{(a_1,a_2)\ell}{\nabla} v(k) = u(k) \text{ and } v(k) = \underset{(a_1,a_2)\ell}{\overset{-1}{\nabla}} u(k) = \frac{e(k_\ell)}{1-\sum_{j=1}^{m} a_j e(j\ell_\ell)}.$$

Substituting $v(k)$ and $u(k)$ in (11) gives (14).

The following example is a numerical verification for (14). □

**Example 2.9.** *Taking* $k = 5, m = 3, n = 3, a_1 = a_2 = a_3 = \ell = 1$ *in* (14) *and using*

$F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 4,$*we have*

$$\frac{e(5_1) - 7e(9_1) - 6e(10_1) - 4e(11_1)}{1 - e(1_1) - e(2_1) - e(3_1)} = \sum_{i=0}^{3} F_i e((5+i)_1)$$

$$\frac{32 - 7(512) - 6(1024) - 4(2048)}{-13} = F_0 e(5_1) + F_1 e(6_1) + F_2 e(7_1) + F_3 e(8_1) = 1376.$$

**Theorem 2.10.** *Let* $1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell) \neq 0$. *Then, an exact solution of the higher order difference equation* $v(k) - \sum_{i=1}^{m} a_i v(k+m\ell) = k^N e((sk)_{s\ell})$ *is given by*

$$\underset{(a)\ell}{\overset{-1}{\nabla}} k^N e((sk)_{s\ell}) = \frac{k^N e((sk)_{s\ell})}{1-\sum_{j=1}^{m} a_j e((j\ell)_\ell)} + \sum_{i=1}^{N} \underset{(a)\ell}{\overset{-1}{\nabla}} \frac{Nc_i k^{N-i} \ell^i \sum_{j=1}^{m} j^i a_j e((js\ell)_{s\ell})}{1-\sum_{j=1}^{m} a_j e((j\ell)_\ell)}. \tag{15}$$

*Proof.* We give proof by induction method. When $m = 0$, by

taking $v(k) = k^0 e((sk)_{s\ell})$ in (7), we get

$$\underset{(a)\ell}{\overset{-1}{\nabla}} k^0 e((sk)_{s\ell}) = \frac{k^0 e((sk)_{s\ell})}{1-\sum_{j=1}^{m} a_j e((j\ell)_\ell)}. \tag{16}$$

When $m = 1$, taking $v(k) = \dfrac{ke((sk)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)}$ in (7), we obtain

$$\underset{(a)\ell}{\nabla} \frac{ke((sk)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)} = \frac{ke((sk)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)} - \frac{a_1(k+\ell)e((s(k+\ell))_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)}$$
$$- \frac{a_2(k+2\ell)e(s(k+2\ell)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)},$$

which is the same as

$$\underset{(a)\ell}{\nabla} \frac{ke((sk)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell_\ell))} = ke((sk)_{s\ell}) - \frac{\ell e((sk)_{s\ell}) \sum_{j=1}^{m} a_j e((js\ell)_{s\ell}) k^0}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)}. \tag{17}$$

Similarly, by taking $v(k) = \dfrac{k^2 e((sk)_{s\ell})}{1 - \sum_{i=1}^{m} a_j e((j\ell)_\ell)}$ in (7), we find

$$\underset{(a)\ell}{\nabla} \frac{k^2 e((sk)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)} = \frac{k^2 e((sk)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)} - \frac{a_1(k+\ell)^2 e(s(k+\ell)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)}$$
$$- \frac{a_2(k+2\ell)^2 e(s(k+2\ell)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)}.$$

By expanding and grouping the terms, we arrive

$$\underset{(a)\ell}{\nabla} \frac{k^2 e((sk)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)} = k^2 e((sk)_{s\ell}) - \frac{2k\ell e((sk)_{s\ell}) \sum_{j=1}^{m} j a_j e((s\ell)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)}$$
$$- \frac{\ell^2 e((sk)_{s\ell}) \sum_{j=1}^{m} j^2 a_j e((js\ell)_{s\ell}) k^0}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)}. \tag{18}$$

For $v(k) = \dfrac{k^3 e((sk)_{s\ell})}{1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)}$, the corresponding relation is obtained as below:

$$\underset{(a)\ell}{\nabla} \frac{k^3 e((sk)_{s\ell})}{1-\sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)} = \frac{k^3 e((sk)_{s\ell})}{1-\sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)} - \frac{a_1 (k+\ell)^3 e(s(k+\ell)_{s\ell})}{1-\sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)}$$

$$- \frac{a_2 (k+2\ell)^3 e(s(k+2\ell)_{s\ell})}{1-\sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)}$$

$$= k^3 e((sk)_{s\ell}) - \frac{3k^2\ell e((sk)_{s\ell}) \sum\limits_{j=1}^{m} j a_j e((js\ell)_{s\ell})}{1-\sum\limits_{j=1}^{m} j a_j e((j\ell)_\ell)}$$

$$- \frac{3k\ell^2 e((sk)_{s\ell}) \sum\limits_{j=1}^{m} j^2 a_j e((js\ell)_{s\ell})}{1-\sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)} - \frac{\ell^3 e((sk)_{s\ell}) \sum\limits_{j=1}^{m} j^3 a_j e((js\ell)_{s\ell}) k^0}{1-\sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)}$$

$$\underset{(a)\ell}{\nabla} \frac{k^3 e((sk)_{s\ell})}{1-\sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)} = k^3 e((sk)_{s\ell}) - \sum_{i=1}^{3} \frac{3c_i k^{3-i}\ell^i \sum\limits_{j=1}^{m} j^i a_j e((js\ell)_{s\ell}) e((sk)_{s\ell})}{1-\sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)}. \quad (19)$$

In general, we find that, by induction,

$$\underset{(a)\ell}{\nabla} \frac{k^N e((sk)_{s\ell})}{1-\sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)} = k^N e((sk)_{s\ell}) - \sum_{i=1}^{N} \frac{N c_i k^{N-i}\ell^i \sum\limits_{j=1}^{m} j^i a_j e((js\ell)_{s\ell}) e((sk)_{s\ell})}{1-\sum\limits_{j=1}^{m} a_j e((j\ell)_\ell)}.$$

Applying $\overset{-1}{\underset{(a)\ell}{\nabla}}$ on both sides, we get (15). $\square$

**Example 2.11.** *Let* $1-\sum\limits_{j=1}^{m} a_j e((js\ell)_\ell) \neq 0$. *Then, by taking* $N=3$ *in* (15),

$$\overset{-1}{\underset{(a)\ell}{\nabla}} k^3 e((sk)_{s\ell}) = \frac{k^3 e((sk)_{s\ell})}{1-\sum\limits_{j=1}^{m} a_j e((js\ell)_\ell)} + \frac{3k^2\ell e((sk)_{s\ell}) \sum\limits_{j=1}^{m} j a_j e((js\ell)_\ell)}{\left(1-\sum\limits_{j=1}^{m} a_j e((js\ell)_\ell)\right)^2}$$

$$+\frac{6k\ell^2 e((sk)_{s\ell})\left(j\sum_{j=1}^{m}a_je((js\ell)_\ell)\right)^2}{\left(1-\sum_{j=1}^{m}a_je((js\ell)_\ell)\right)^3}+\frac{6\ell^3 e((sk)_{s\ell})\left(\sum_{j=1}^{m}a_je((js\ell)_\ell)\right)^3}{\left(1-\sum_{j=1}^{m}a_je((js\ell)_\ell)\right)^4}$$

$$+\frac{3k\ell^2 e((sk)_{s\ell})\left(j^2\sum_{j=1}^{m}a_je((js\ell)_\ell)\right)}{\left(\sum_{j=1}^{m}a_je((js\ell)_\ell)\right)^2}$$

$$+\frac{6\ell^3 e((sk)_{s\ell})\left(\sum_{j=1}^{m}a_je((js\ell)_\ell)\right)\left(\sum_{j=1}^{m}j^2a_je((js\ell)_\ell)\right)}{\left(1-\sum_{j=1}^{m}a_je((js\ell)_\ell)\right)^3}+\frac{\ell^3 e((sk)_{s\ell})\left(\sum_{j=1}^{m}j^3a_je((js\ell)_\ell)\right)}{\left(1-\sum_{j=1}^{m}a_je((js\ell)_\ell)\right)^2} \tag{20}$$

*Now (20) is an exact solution of the higher order difference equation* $\underset{(a)\ell}{\nabla} v(k)=k^3e(k_\ell)$.

*Proof.* Taking N=3 in (15), we get the relation

$$\underset{(a)\ell}{\overset{-1}{\nabla}} k^3e((sk)_{s\ell})=\frac{k^3e((sk)_{s\ell})}{1-\sum_{j=1}^{m}a_je(js\ell_\ell)}+\sum_{i=1}^{3}\underset{(a)\ell}{\overset{-1}{\nabla}}\frac{3c_ik^{3-i}\ell^i\sum_{j=1}^{m}j^ia_je((js\ell)_\ell)e((sk)_{s\ell})}{1-\sum_{j=1}^{m}a_je((js\ell)_\ell)}$$

$$=\frac{k^3e((sk)_{s\ell})}{1-\sum_{j=1}^{m}a_je((js\ell)_\ell)}+\frac{3\ell\sum_{j=1}^{m}a_je((js\ell)_\ell)}{1-\sum_{j=1}^{m}a_je((js\ell)_\ell)}\underset{(a)\ell}{\overset{-1}{\nabla}}k^2e((sk)_{s\ell})$$

$$+\frac{3\ell^2\sum_{j=1}^{m}a_je((j^2s\ell)_\ell)}{1-\sum_{j=1}^{m}a_je((js\ell)_\ell)}\underset{(a)\ell}{\overset{-1}{\nabla}}ke((sk)_{s\ell})+\frac{\ell^3\sum_{j=1}^{m}j^3a_je((js\ell)_\ell)}{1-\sum_{j=1}^{m}a_je((js\ell)_\ell)}\underset{(a)\ell}{\overset{-1}{\nabla}}k^0e((sk)_{s\ell}). \tag{21}$$

Applying $\overset{-1}{\underset{(a)\ell}{\nabla}}$ on both sides of (17) and using (16),we find that

$$\overset{-1}{\underset{(a)\ell}{\nabla}} ke((sk)_{s\ell}) = \frac{ke((sk)_{s\ell})}{\left(1-\sum_{j=1}^{m} a_j e((js\ell)_\ell)\right)} + \frac{\ell\left(\sum_{j=1}^{m} ja_j e((js\ell)_\ell)\right) e((sk)_{s\ell})}{\left(1-\sum_{j=1}^{m} a_j e((js\ell)_\ell)\right)^2}. \tag{22}$$

Applying $\overset{-1}{\underset{(a)\ell}{\nabla}}$ on both sides of (18) and using (22),(16),we arrive

$$\overset{-1}{\underset{(a)\ell}{\nabla}} k^2 e((sk)_{s\ell}) = \frac{k^2 e((sk)_{s\ell})}{\left(1-\sum_{j=1}^{m} a_j e((js\ell)_\ell)\right)} + \frac{2k\ell\left(\sum_{j=1}^{m} ja_j e((js\ell)_\ell)\right) e((sk)_{s\ell})}{\left(1-\sum_{j=1}^{m} a_j e((js\ell)_\ell)\right)^2}$$

$$+\frac{2\ell^2\left(\sum_{j=1}^{m} ja_j e((js\ell)_\ell)\right)^2 e((sk)_{s\ell})}{\left(1-\sum_{j=1}^{m} a_j e((js\ell)_\ell)\right)^3} + \frac{\ell^2\left(\sum_{j=1}^{m} j^2 a_j e((js\ell)_\ell)\right) e((sk)_{s\ell})}{\left(1-\sum_{j=1}^{m} a_j e((js\ell)_\ell)\right)^2}. \tag{23}$$

Substituting the above value in (21) and using (16),(22),(23),we obtain (20).

□

**Corollary 2.12.** *(Higher order Fibonacci formula for product of polynomial and extorial function).If* $v(k) = \overset{-1}{\underset{(a)\ell}{\nabla}} k^N e((sk)_{s\ell})$ *is as given in* (15)*, then we have*

$$\overset{-1}{\underset{(a)\ell}{\nabla}} k^N e((sk)_{s\ell}) - \sum_{j=1}^{m}\sum_{i=j}^{m} a_i F_{n-(i-j)} \overset{-1}{\underset{(a)\ell}{\nabla}} (k+(n+j)\ell)^N e(s(k+(n+j)\ell)_{s\ell})$$

$$= \sum_{i=0}^{n} F_i (k+i\ell)^N e(s(k+i\ell)_{s\ell}). \tag{24}$$

*Proof.* Taking $u(k) = k^N e(sk_{s\ell})$ in Theorem 2.7 gives (24). □

**Example 2.13.** *By taking* $k = 5$*,*$N = 3$*,*$n = 2$*,*$m = 3$*,*$a_1 = a_2 = a_3 = 1$ *and* $s = \ell = 1$ *in* (24)*, we find that*

$$\overset{-1}{\underset{(1,1,1)}{\nabla}} (5)^3 e((5)_1) - 4 \overset{-1}{\underset{(1,1,1)}{\nabla}} (8)^3 e((8)_1) - 3 \overset{-1}{\underset{(1,1,1)}{\nabla}} (9)^3 e((9)_1) - 2 \overset{-1}{\underset{(1,1,1)}{\nabla}} (10)^3 e((10)_1)$$

$$= \sum_{i=0}^{2} F_i(5+i\ell)^3 e((5+i)_1) = 105632.$$

*The following is the product formula for inverse of higher order Fibonacci operator.*

**Theorem 2.14.** *Let $u(k)$ and $v(k)$ be the real valued function. Then we have*

$$\underset{(a)\ell}{\overset{-1}{\nabla}} [u(k)\, v(k)] = u(k) \underset{(a)\ell}{\overset{-1}{\nabla}} v(k) - \sum_{i=1}^{m} a_i \underset{(a)\ell}{\overset{-1}{\nabla}} \left[ \underset{(a)\ell}{\overset{-1}{\nabla}} v(k+i\ell) \underset{(e_i)}{\nabla} u(k) \right], \tag{25}$$

*where $e_i = (0,0,0,\cdots,1,0,0,\cdots,0)$, $i^{th}$ components is 1 other component are zero.*

*Proof.* Taking $v(k) = u(k)\, w(k)$ in (7), we get

$$\underset{(a)\ell}{\nabla} [u(k)\, w(k)] = u(k)\, w(k) - a_1 u(k+\ell)\, w(k+\ell) - a_2 u(k+2\ell)\, w(k+2\ell)$$
$$- \cdots - a_m u(k+m\ell)\, w(k+m\ell)$$

Adding and Subtracting $a_1 u(k)\, w(k+\ell)$, $a_2 u(k)\, w(k+2\ell)$,

$\cdots, a_m u(k+m\ell)\, w(k+m\ell)$ gives

$$\underset{(a)\ell}{\nabla} [u(k)w(k)] = u(k) \underset{(a)\ell}{\nabla} w(k) + \sum_{i=1}^{m} a_i w(k+i\ell) \underset{e_i}{\nabla} u(k)$$

Taking $w(k) = \underset{(a)\ell}{\overset{-1}{\nabla}} v(k)$ and applying $\underset{(a)\ell}{\overset{-1}{\nabla}}$ on both sides, we get (25),

where $e_i = (0,0,0.\cdots,1,0,0.)$ and $a = (a_1, a_2, \cdots, a_m)$. □

**Corollary 2.15.** *Let u(k) and v(k) be the real valued functions. Then,*

$$\underset{(a)\ell}{\overset{-1}{\nabla}} u(k)e(k_\ell) = u(k) \underset{(a)\ell}{\overset{-1}{\nabla}} e(k_\ell) - \sum_{i=1}^{m} a_i \underset{(a)\ell}{\overset{-1}{\nabla}} \left[ \underset{(a)\ell}{\overset{-1}{\nabla}} e((k+i\ell)_\ell) \underset{e_i}{\nabla} u(k) \right] \tag{26}$$

*Proof.* Taking $v(k) = e(k_\ell)$ in (25), we get (26) □

**Corollary 2.16.** *An exact solution of the higher order diffference equation $v(k) - \sum_{i=1}^{m} a_i v(k+i\ell) = k^2 e(k_\ell)$ is given by*

$$\underset{(a)\ell}{\overset{-1}{\nabla}} k^2 e(k_\ell) = \frac{k^2 e(k_\ell)}{\left(1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)\right)} + \frac{2k\ell \sum_{j=1}^{m} j a_j e((k+(j\ell))_\ell)}{\left(1 - \sum_{j=1}^{m} a_j e((j\ell)_\ell)\right)^2}$$

$$+\frac{2\ell^2\sum_{i=1}^{m} ia_i\sum_{j=1}^{m} ja_j e((k+(i+j)\ell)_\ell)}{\left(1-\sum_{j=1}^{m} a_j e((j\ell)_\ell)\right)^3}+\frac{\sum_{j=1}^{m} i^2 a_i\ell^2 e((k+i\ell)_\ell)}{\left(1-\sum_{j=1}^{m} a_j e(j\ell_\ell)\right)^2}. \tag{27}$$

*Proof.* Taking $u(k)=k$ in (26) and using (10), we find

$$\underset{(a)\ell}{\overset{-1}{\nabla}} ke(k_\ell)=\frac{ke(k_\ell)}{\left(1-\sum_{j=1}^{m} a_j e((j\ell)_\ell)\right)}+\frac{\sum_{j=1}^{m} ja_j\ell e((k+(j\ell))_\ell)}{\left(1-\sum_{j=1}^{m} a_j e(j\ell_\ell)\right)^2}. \tag{28}$$

Taking $u(k)=k^2$ in (26) and using (10) yield

$$\underset{(a)\ell}{\overset{-1}{\nabla}} k^2e(k_\ell)=\frac{k^2e(k_\ell)}{\left(1-\sum_{j=1}^{m} a_j e((j\ell)_\ell)\right)}+\frac{2\sum_{j=1}^{m} ja_j\ell}{\left(1-\sum_{j=1}^{m} a_j e(j\ell)_\ell\right)}\underset{(a)\ell}{\overset{-1}{\nabla}} ke((k+i\ell)_\ell)$$

$$\frac{\sum_{j=1}^{m} \ell^2 j^2 a_j}{\left(1-\sum_{j=1}^{m} a_j e((j\ell)_\ell)\right)}\underset{(a)\ell}{\overset{-1}{\nabla}} k^0 e((k+i\ell)_\ell).$$

By substituting (28) in the above and using (10),we get the proof of (27) □

**Corollary 2.17.** *If* $v(k)=\underset{(a)\ell}{\overset{-1}{\nabla}} k^2e(k_\ell)$ *is an exact solution given by* (27)*, then the higher order Fibonacci summation formula for* $k^2e(k_\ell)$ *is obtained as*

$$\underset{(a)\ell}{\overset{-1}{\nabla}} k^2e(k_\ell)-\sum_{j=1}^{m}\sum_{i=j}^{m} a_iF_{n-(i-j)}\underset{(a)\ell}{\overset{-1}{\nabla}}(k+(n+j)\ell)^2e((k+(n+j)\ell)_\ell)$$

$$=\sum_{i=0}^{n} F_i(k+i\ell)^2e((k+i\ell)_\ell). \tag{29}$$

*Proof.* by taking $u(k)=k^2e(k_\ell)$ in Theorem 2.7,we get (29) □

**Example 2.18.** *Let* $k=8$,$n=2$,$m=3$,$a_1=a_2=a_3=1$ *in* (29)*.Then*

$$\underset{(1,1,1)}{\overset{-1}{\nabla}} 8^2e(8_1)-4\underset{(1,1,1)}{\overset{-1}{\nabla}} 11^2e(11_1)-3\underset{(1,1,1)}{\overset{-1}{\nabla}} 12^2e(12_1)-2\underset{(1,1,1)}{\overset{-1}{\nabla}} 13^2e(13_1)$$

$$= \sum_{i=0}^{2} F_i(8+i)^2 e((8+i)_1) = 262656.$$

## 3. Conclusion

The newly arrised extorial function is applied to obtain solution of certain type of higher order difference equation involving Fibonacci difference operator. Several results on sum of finite series are derived by the inverse of Fibonacci nabla difference operator. Several applications in life science can be obtained by this fibonacci nabla operator and replacing exponential into extorial functions.

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